The Symplectic Structure of Curves in three dimensional spaces of constant curvature and the equations of mathematical physics

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1 Introduction

This paper defines a symplectic form on the infinite dimensional Fréchet manifold of framed curves of fixed length over a three dimensional simply connected Riemannian manifold of constant curvature. The framed curves are anchored at the initial point and are further constrained by the condition that the tangent vector of the projected curve coincides with the first leg of the orthonormal frame. Such class of curves are called anchored Darboux curves and in particular include the Serret-Frenet framed curves.

The symplectic form ω is defined on the universal covers of the orthonormal frame bundles of the underlying manifolds: $SL_2(C)$ for the hyperboloid $\mathbb{H}^3$, $SU_2 \times SU_2$ for the sphere S^3, and the semidirect product $E^3 \rhd SU_2$ for the Euclidean space E^3. The form ω is left invariant on each of the above groups, and is induced by the Poisson-Lie bracket on the appropriate Lie algebra. More precisely, the form ω in each of the above non-Euclidean cases is defined on the Cartan space $\mathfrak{p}$ corresponding to the decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$$

of the Lie algebra $\mathfrak{g}$ subject to the usual Lie algebraic relations

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, \ [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, \ [\mathfrak{k}.\mathfrak{k}] = \mathfrak{k}$$

In the case of the hyperboloid $\mathfrak{g}$ is equal to $sl_2(C)$ and the Cartan space is equal to the space of the Hermitian matrices $\mathfrak{h}$, and in the case of the sphere $\mathfrak{g}$ is equal to $su_2 \times su_2$ and the Cartan space is isomorphic to the space of skew-Hermitian matrices $\mathfrak{k}$. In each case the maximal compact subalgebra $\mathfrak{k}$ is equal to the Lie algebra of skew Hermitian matrices. The symplectic forms in each of these two cases are isomorphic

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to each other as a consequence of the isomorphism between $\mathfrak{k}$ and $\mathfrak{h}$ given by $i\mathfrak{h} = \mathfrak{k}$. The Euclidean space E^3 is identified with $\mathfrak{h}$ equipped with the metric defined by the trace form, and its framed curves are represented in the semidirect product $\mathfrak{h} \triangleright \mathfrak{k}$. The Euclidean Darboux curves inherit the hyperbolic symplectic form ω which is isomorphic to the symplectic form used by J. Millson and B. Zombro in ([17]).

Each group G mentioned above is a principal SU_2 bundle over the underlying symmetric space with a natural connection defined by the left invariant vector fields that take values in the Cartan space $\mathfrak{p}$. The vertical distribution is defined by the left invariant vector fields that take their values in $\mathfrak{k}$. In this setting then, anchored Darboux curves are the solutions $g(s) \in G$ of a differential equation

$$\frac{dg}{ds}(s) = g(s)(E_1 + u_1(s)A_1 + u(s)A_2 + u_3(s)A_3) \tag{1}$$

with $g(0) = I$, where E_1 is a fixed unit vector in the Cartan space $\mathfrak{p}$. The matrices A_1, A_2, A_3 denote the skew-Hermitian Pauli matrices, and $u_1(s), u_2(s), u_3(s)$ are arbitrary real valued functions on a fixed interval $[0, L]$. Each anchored Darboux curve defines a horizontal Darboux curve $h(s) \in G$ that is a solution of the differential equation

$$\frac{dh}{ds}(s) = h(s)\Lambda(s), \ \Lambda(s) = R(s)E_1R^{-1}(s) \tag{2}$$

with $R(s)$ the solution curve in SU_2 of the equation

$$\frac{dR}{ds} = R(s)(u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3) \tag{3}$$

that satisfies $R(0) = I$. The symplectic form for the hyperbolic Darboux curves is given by

$$\omega_\Lambda(V_1, V_2) = \frac{1}{i} \int_0^L \langle \Lambda(s), [U_1(s), U_2(s)] \rangle \, ds \tag{4}$$

with $U_1(s)$ and $U_2(s)$ Hermitian matrices orthogonal to the tangent vector $\Lambda(s)$, that further satisfy $U_j(0) = 0$ and $\frac{dV_j}{ds}(s) = U_j(s)$ for $j = 1, 2$.

In the spherical case the symplectic form has the same form as in the hyperbolic case, except for the factor $\frac{1}{i}$, which is omitted. The matrices U_j in this case take values in $\mathfrak{k}$ and satisfy

$$\frac{dV_j}{ds}(s) = [\Lambda(s), V_j(s)] + U_j(s)$$

for $j = 1, 2$.

The second part of the paper is devoted to the Hamiltonian flow associated with the function

$$f(g(s)) = \frac{1}{2} \int_0^L ||\frac{d\Lambda}{ds}(s)||^2 \, ds = \frac{1}{2} \int_0^L \kappa^2(s) \, ds$$

where g denotes a frame-periodic horizontal Darboux periodic curve, i.e., a Darboux curve for which the solution $R(s)$ of equation (3) is periodic . Here $\kappa(s)$ denotes the curvature of the projected curve $x(s)$ in the underlying symmetric space.

On the level of Lie algebras, the Hamiltonian flow induced by the symplectic form ω generates Heisenberg's magnetic equation in the Cartan space $\mathfrak{p}$ given by

$$\tfrac{\partial \Lambda}{\partial t}(s,t) = \tfrac{1}{i}[\Lambda(s), \tfrac{\partial^2 \Lambda}{\partial s^2}(s,t)] \tag{5}$$

in the hyperbolic and the Euclidean case, and by

$$\frac{\partial \Lambda}{\partial t}(s,t) = [\Lambda(s), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)]$$

in the spherical case.

The corresponding frame $R(s,t)$ defines a complex function

$$\psi(s,t) = u(s,t)\exp(i\int_0^s u_1(x,t)\,dx) \tag{6}$$

with $u(s,t) = u_2(s,t) + iu_3(s,t)$ that is a solution of the non-linear Schroedinger's equation

$$-i\tfrac{\partial \psi}{\partial t}(t,s) = \tfrac{\partial^2 \psi}{\partial s^2}(t,s) + 1/2|\psi(t,s)|^2(\psi(t,s)+c) \tag{7}$$

(Theorem 9).

This finding clarifies a remarkable observation of H. Hasimoto ([8]) that the function

$$\psi(s,t) = \kappa(s,t)\exp(i\int_0^s \tau(x,t)\,dx)$$

where $\kappa(t,s)$ and $\tau(t,s)$ are the curvature and the torsion of a curve $\gamma(t,s))$ that evolves according to to the filament equation

$$\frac{\partial \gamma}{\partial t}(t,s) = \kappa(t,s)B(t,s) \tag{8}$$

is a solution of the non-linear Schroedinger equation

$$-i\frac{\partial \psi}{\partial t}(t,s) = \frac{\partial^2 \psi}{\partial s^2}(t,s) + 1/2|\psi(t,s)|^2(\psi(t,s)+c)$$

for some complex constant c. Indeed, when the frame $R(s)$ in equation (3) is a Serret-Frenet frame then ψ given by (6) coincides with Hasimoto's function up to a unitary constant.

The curves that correspond to the critical points of $f = \tfrac{1}{2}\int_0^L \kappa^2(s)\,ds$ are called elastic. The material in Section 5 shows that the elastic curves with periodic curvatures always generate periodic soliton solutions for the non-linear Schroedinger's equation. The extremal equations associated with elastic curves, obtained through the Hamiltonian formalism of Lie groups, provide for easy transitions to solitons:

The Hamiltonian system associated with elastic curves is completely integrable having four integrals of motion: the Hamiltonian itself, which may be regarded as the energy of the elastic curve, two Casimirs, and another integral of motion H_1 due

to an extra symmetry in the system. The elastic curves that generate solitons reside on a fixed energy level and propagate with the speed equal to H_1. The fact that the equations for the heavy top form an invariant subsystem of the equations for the elastic curves ([10] and [11]) makes the connection between elastic curves and solitons even more intriguing: the speed of the soliton corresponds to the angular momentum along the axis of symmetry for the top of Lagrange.

The formalism of this paper suggests that there is a class of functions $f_0, f_1, f_2 \ldots$ over the space of Darboux curves that begins with $f_0 = \frac{1}{2} \int_0^L \kappa^2(s) \, ds$ having the property that any two functions Poisson commute. It is shown in the paper that f_1 and f_2 given by

$$f_1 = i \int_0^L \langle [\Lambda(s), \frac{d\Lambda}{ds}(s)], \frac{d^2\Lambda}{ds^2}(s) \rangle \, ds, \ f_2 = \int_0^L (||\ddot{\Lambda}(s)||^2 - \frac{5}{4} ||\dot{\Lambda}(s)||^4) \, ds$$

are in this class.

The above funtions can be expressed either in terms of the geometric invariants of the underlying Darboux curve as:

$$f_1 = \int_0^L \kappa^2(s) \tau(s) \, ds, \ f_2 = \int_0^L (\frac{\partial \kappa}{\partial s}(s)^2 + \kappa^2(s) \tau^2(s) - \frac{1}{4} \kappa^4(s)) \, ds$$

in which case they agree with the first three functions on the list presented by J. Langer and R. Perline in ([14]), or they can be expressed in terms of the complex function $u(s)$ defined by equation (6) as $f_0 = \frac{1}{2} \int_0^L |u(s)|^2 \, ds$ and

$$f_1 = \frac{1}{2i} \int_0^L (\bar{u}\dot{u} - u\dot{\bar{u}}) \, ds, \ f_2 = \int_0^L (|\frac{\partial u}{\partial s}(s,t)|^2 - \frac{1}{4} |u(s,t)|^4) \, ds$$

in which case they correspond to the first three conserved quantities, the number of particles, the momentum and the energy, in the paper by C.Shabat and V. Zakharov in ([18]).

The paper is organized as follows. The first part of the paper leads up to the symplectic form. The fact that the sphere S^3 is the same as the unitary group SU_2 makes the presentation slightly more accessible for the sphere than for the hyperboloid; for that reason the paper begins with the geometric preliminaries on SU_2 required for the definition of the symplectic form for the spherical Darboux curves. This material is presented in Section 2. The analogous material for the hyperbolic Darboux curves is presented in Section 3.

The second part of the paper begins with Section4 and is devoted to the Hamiltonian flow corresponding to $f_0 = \frac{1}{2} \int_0^L \kappa^2(s) \, ds$. This section also contains a discussion of the Euclidean symplectic form and its connection to the existing results in the literature. Section 5 deals with elastic curves and the soliton solutions for the non-linear Schroedinger's equation. The final section (Section 6) contains a brief discussion of the hierarchy of Poisson commuting functions and their connections to the hierarchies presented in ([14] and [18]).

2 Darboux curves on the sphere and their symplectic form

2.1 Notations and geometric preliminaries

The three dimensional sphere $S^3 = \{x \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_3^2 + x_4^2 = 1\}$ can be represented either by the unit quaternions or by the matrices $X = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ with $z = x_0 + ix_1$, $w = x_2 + ix_3$ and $|z|^2 + |w|^2 = 1$. The most direct way to these representations is through the identification of points $x = (x_o, x_1, x_2, x_3)$ in $\mathbb{R}^4$ with the matrices

$$X = x_0 I + x_1 E_1 + x_2 E_2 + x_3 E_3 = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

where

$$E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

. Matrices E_1, E_2, E_3 conform to the relations

$$E_1^2 = E_2^2 = E_3^2 = -I,$$

$$E_1 E_2 = E_3, E_3 E_1 = E_2, E_2 E_3 = E_1$$

and can be identified with i, j, k, the standard basis for the imaginary quaternions. The sphere S^3 then is equal to the set of matrices $X = x_0 I + x_1 E_1 + x_2 E_2 + x_3 E_3$ whose determinant $\text{Det}(X) = 1$. Alternatively, the sphere could also be defined as the group of 2×2 matrices g with complex entries whose inverses g^{-1} are equal to their Hermitian transposes g^*. This group is called the special unitary group and is traditionally denoted by SU_2.

However, in this paper the special unitary group shall be denoted by K for additional simplicity in notation. The Lie algebra of K denoted by $\mathfrak{k}$ consists of the matrices

$$A = \begin{pmatrix} ix_1 & u_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}$$

with x_1, x_2, x_3 arbitrary real numbers. The Lie bracket $[A, B]$ is defined by $[A, B] = BA - AB$ for A and B in $\mathfrak{k}$. The tangent vectors at any point X of K shall be represented by matrices XA with $A \in \mathfrak{k}$, and the tangent bundle TK of K shall be represented by the product $K \times \mathfrak{k}$ in terms of the left-invariant vector fields $X \to XA, A \in \mathfrak{k}$.

Definition 2.1 *Matrices*

$$A_1 = \tfrac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{9}$$

shall be known as the Pauli matrices.

The Pauli matrices form a basis for $\mathfrak{k}$ and conform to the following Lie bracket relations:

$$[A_1, A_2] = -A_3, \ [A_1, A_3 = A_2, \ [A_2, A_3] = -A_1 \tag{10}$$

Definition 2.2 *The trace form is a quadratic form on $\mathfrak{k}$ defined by*

$$\langle A, B \rangle = -2Tr(AB) \tag{11}$$

for any matrices A, B in $\mathfrak{k}$, where $Tr(AB)$ denotes the trace of AB.

It follows that

$$\langle A, B \rangle = \sum u_i v_i$$

whenever

$$A = \sum u_i A_i \text{ and } B = \sum v_i A_i.$$

Therefore, the trace form is positive definite on $\mathfrak{k}$, and the matrices A_1, A_2, A_3 form an orthonormal basis in $\mathfrak{k}$. Then,

$$||A|| = \sqrt{\langle A, A \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

for any matrix $A = \sum u_i A_i$.

The trace form, in addition, satisfies these invariance properties:

$$\langle A, [B, C] \rangle = \langle [A, B], C \rangle, \text{ and } \langle gAg^*, gBg^* \rangle = \langle A, B \rangle \tag{12}$$

for any matrices A, B, C in $\mathfrak{k}$, and any g in K.

The Riemannian structure of the sphere, inherited from the Euclidean inner product (x, y) in $\mathbb{R}^4$ is related to the trace form according to the following lemma.

Lemma 1 *Suppose that $\frac{dX_1}{ds}(0) = XA$ and $\frac{dX_2}{ds}(0) = XB$ denote tangent vectors at X represented by the curves $X_1(s)$ and $X_2(s)$ in K. Then,*

$$(\tfrac{dx_1}{ds}(0), \tfrac{dx_2}{ds}(0)) = 4\langle A, B \rangle. \tag{13}$$

Proof. For each matrix $X = x_0 I + x_1 E_1 + x_2 E_2 + x_3 E_3$ the companion matrix $X^\dagger$ is defined by

$$X^\dagger = x_0 I - x_1 E_1 - x_2 E_2 - x_3 E_3 = \begin{pmatrix} x_0 - ix_1 & -x_2 - ix_3 \\ x_2 - ix_3 & x_0 + ix_1 \end{pmatrix}$$

Evidently, $X^\dagger = -X$ when X belongs to $\mathfrak{k}$, and

$$\tfrac{1}{2}(XY^\dagger + YX^\dagger) = (x, y)I, \tag{14}$$

as can be verified by an easy calculation.

Therefore,

$$
\begin{aligned}
(\frac{dx_1}{ds}(0), \frac{dx_2}{ds}(0))I &= \frac{1}{2}((\frac{dX_1}{ds}(0))(\frac{dX_2}{ds}(0))^\dagger + (\frac{dX_2}{ds}(0))(\frac{dX_1}{ds}(0))^\dagger) \\
&= \frac{1}{2}((XA)(XB)^\dagger + (XB)(XA)^\dagger) = \frac{1}{2}(AB^\dagger + BA^\dagger) \\
&= -\frac{1}{2}(AB + BA) = 4\langle A, B \rangle
\end{aligned}
$$

■

Definition 2.3 *The group K shall be considered a Riemannian manifold with its Riemannian metric given by the trace form.*

It follows that the length $l(X)$ of any curve $X(s)$, $s \in [0, L]$ is given by

$$
l(X) = \int_0^L ||\Lambda(s)|| \, ds,
$$

where $\Lambda(s) = X(s)^* \frac{dX}{ds}(s)$. The length $l(X)$ is equal to half the length of the Euclidean metric in $\mathbb{R}^4$ given by the usual formula $\int_0^L (\frac{dx}{ds}(s), \frac{dx}{ds}(s))^{\frac{1}{2}} \, ds$

Definition 2.4 *The covariant derivative of a curve of tangent vectors $v(s)) = X(s)B(s)$ along a curve $X(s)$ is defined by*

$$
\frac{D_X}{ds}(V)(s) = X(s)(\frac{dB}{ds}(s) + \frac{1}{2}[A(s), B(s)])
$$

where $A(s) = X^*(s)\frac{dX}{ds}(s)$.

The reader can easily verify that the covariant derivative is equal to the orthogonal projection of the ordinary derivative in $\mathbb{R}^4$ onto the tangent space od the sphere when the sphere is considered a subset of $\mathbb{R}^4$.

Lemma 2 *Suppose that $X(s, t)$ is a field of curves in K with the infinitesimal directions*

$$
A(s, t) = X^*(s, t)\frac{\partial X}{\partial s}(s, t) \text{ and } B(s, t) = X^*(s, t)\frac{\partial X}{\partial t}(s, t)
$$

Then,

$$
\frac{\partial A}{\partial t}(s, t) - \frac{\partial B}{\partial s}(s, t) + [A(s, t), B(s, t)] = 0 \tag{15}
$$

Proof. As in any Riemannian manifold,

$$
\frac{D_X}{ds}(\frac{\partial X}{\partial t})(s, t) = \frac{D_X}{dt}(\frac{\partial X}{\partial s})(s, t)
$$

Therefore,

$$X(s,t)(\frac{\partial B}{\partial s}(s,t) + \frac{1}{2}[B(s,t), A(s,t)]) = X(s,t)(\frac{\partial A}{\partial t}(s,t) + \frac{1}{2}[A(s,t), B(s,t)])$$

which implies that

$$\frac{\partial A}{\partial t}(s,t) - \frac{\partial B}{\partial s}(s,t) + [A(s,t), B(s,t)] = 0$$

■

This equation is known as the zero-curvature equation ([5]).

2.2 The orthonormal frame bundle and the Darboux curves

The orthonormal frame bundle of K is the totality of pairs (X,F) where X is a point in K and where $F = (v_1, v_2, v_3)$ is an orthonormal frame at X. Curves in the orthonormal frame bundle of K shall be called framed curves. Basic to this paper is a set of framed curves defined by the following properties:

(a) Each curve is defined over a fixed interval [0,L], and conforms to the fixed initial condition

$$X(0) = I, F(0) = (v_1(0), v_2(0), v_3(0)) = (A_1, A_2, A_3).$$

(b) For each curve $(X(s), F(s))$ the orthonormal frame $F(s) = (v_1(s), v_2(s), v_3(s))$ defined along $X(s)$ is adapted to the curve $X(s)$ by the requirement that $v_1(s) = \frac{dX}{ds}$ for all $s \in [0, L]$.

Definition 2.5 *Framed curves that satisfy conditions (a) and (b) above shall be called anchored Darboux curves.*

For each Darboux curve $(X(s), F(s))$ condition (b) implies that $||\frac{dX}{ds}(s)|| = 1$, which then implies that L is the length of the projected curve curve X.

Darboux curves can be embedded in the space of curves in $K \times K$, the universal cover of the orthonormal frame bundle of K through the following realization. Each pair (p,q) in $K \times K$ defines an orthonormal frame (v_1, v_2, v_3) at $X = pq^*$ where

$$v_1 = pA_1q^* = pq^*(qA_1q^*), v_2 = pA_2q^* = pq^*(qA_2q^*), v_3 = pA_3q^* = pq^*(qA_3q^*)$$

Conversely, every orthonormal frame at a point $X \in K$ can be represented by the tangent vectors $v_1 = XB_1, v_2 = XB_2, v_3 = XB_3$ for some matrices B_1, B_2, B_3 in $\mathfrak{k}$ that are orthonormal relative to the trace form. Then, there is matrix $q \in K$ such that

$$v_1 = qA_1q^*, v_1 = qA_2q^*, v_1 = qA_3q^*$$

There are exactly two matrices $\pm q$ which satisfy the preceding equalities. Having found q, p is uniquely defined by $p = Xq$.

The fact that $K \times K$ is a double cover of the orthonormal frame bundle of the sphere does not matter for the subsequent exposition since Darboux curves are anchored at the identity.

It follows from above that anchored Darboux curves can be represented by curves $(p(s), q(s))$ in $K \times K$ that are the solutions of

$$\tfrac{dp}{ds} = p(s)P(s), \ \tfrac{dq}{ds} = q(s)Q(s) \tag{16}$$

subject to the conditions that $p(0) = I$, $q(0) = I$, and that $P(s) - Q(s) = q(s)A_1q^*(s)$. The last condition reflects the fact that the curve of frames

$$(X(s)q(s)A_1q^*(s), X(s)q(s)A_2q^*(s), X(s)q(s)A_3q^*(s))$$

along the curve $X(s)$ in K is adapted to the tangent vector $X(s)A(s)$ of $X(s)$ through the relation $X(s)A(s) = X(s)q(s)A_1q^*(s)$, which implies that $A(s) = q(s)A_1q^*(s)$.

Therefore anchored Darboux curves can be redefined as pairs of curves $(X(s), R(s))$ in $K \times K$ that are the solutions of the initial value problem

$$\tfrac{dX}{ds}(s) = X(s)\Lambda(s), \ \tfrac{dR}{ds}(s) = R(s)A(s) \tag{17}$$

,

$$X(0) = R(0) = I, \ \text{subject to } \Lambda(s) = R(s)A_1R^*(s) \tag{18}$$

.

Definition 2.6 *The projection $X(s)$ of an anchored Darboux curve shall be called horizontal Darboux curve.*

The horizontal Darboux curves are parametrized by three arbitrary functions $u_1(s), u_2(s), u_3(s)$ (given by $A(s) = \sum u_i(s)A_i$ in (18)). These functions are related to the curvature $\kappa(s)$ and the torsion $\tau(s)$ of the projected curve $X(s)$ through the following equations:

$$\kappa(s) = u_3(s)\cos\theta(s) - u_2(s)\sin\theta(s), \ \frac{d\theta}{ds}(s) + u_1(s) = \tau(s)$$

subject to the constraint

$$u_2(s)\cos\theta(s) + u_3(s)\sin\theta(s) = 0$$

The angle θ represents the rotation of the Darboux frame relative to the Serret-Frenet frame in the plane perpendicular to the tangent vector of the projected curve.

Recall that the Serret-Frenet frame $(v_1(s), v_2(s), v_3(s))$, generated by a curve $X(s)$, is defined by the following relations: $\frac{dX}{ds} = v_1$ and

$$\tfrac{D_X}{ds}(v_1) = \kappa v_2, \ \tfrac{D_X}{ds}(v_2) = -\kappa v_1 + \tau v_3, \ \tfrac{D_X}{ds}(v_3) = -\tau v_2 \tag{19}$$

In the representation of the frame bundle as $K \times K$, the Serret-Frenet frame is represented by the matrix $R(s)$ through the relations

$$v_1(s) = X(s)(R(s)A_1R^*(s)), v_2(s) = X(s)(R(s)A_2R^*(s)), v_3(s) = X(s)(R(s)A_3R^*(s))$$

It now follows from Definition 2.4 that

$$\begin{aligned} \frac{D_X}{ds}(v_1) &= X(R[A_1, A]R^*) = -u_3X(RA_3R^*) + u_2X(RA_2R^*) \\ &= u_3v_2 - u_3v_3 \end{aligned}$$

from which it follows that $u_3 = \kappa$ and $u_2 = 0$. Furthemore,

$$\begin{aligned} \frac{D_X}{ds}(v_2)(s) &= X(s)R(s)([A_2, A(s)] + \frac{1}{2}[A_1, A_2])R^*(s)) \\ &= -u_3(s)X(s)(R(s)A_1R^*(s)) + (u_1(s) - \frac{1}{2})X(s)(R(s)A_3R^*(s)) \\ &= -u_3(s)v_1(s) + (u_1(s) - \frac{1}{2})v_3(s) \end{aligned}$$

implies that $\tau(s) = u_1(s) - \frac{1}{2}$.

Therefore the frame curve $R(s)$ corresponds to the Serret-Frenet frame whenever $u_2(s) = 0$, in which case $u_3(s) = \kappa(s)$ and $u_1(s) = \tau(s) + \frac{1}{2}$.

More generally,

Theorem 1 *(a) For any solution $(X(s), R(s))$ of equations (17) and (18) with $A(s) = \sum u_i(s)A_i$,*

$$\kappa^2(s) = u_2^2(s) + u_3^2(s)$$

where $\kappa(s)$ denotes the curvature of the base curve $X(s)$.

(b) Every curve $X(s)$ which satisfies $||\frac{dX}{ds}(s)|| = 1$, $X(0) = I$, and $\frac{dX}{ds}(0) = A_1$ is the projection of an anchored Darboux curve.

Proof.

$$\begin{aligned} \kappa^2(s) &= ||\frac{D_X}{ds}(\frac{dX}{ds})(s)||^2 \\ &= ||X(s)(\frac{d}{ds}(R(s)A_1R^*(s))||^2 = ||R(s)[A_1, A(s)]R^*(s)||^2 \\ &= || - u_2(s)A_3 + u_3(s)A_2||^2 = u_2^2(s) + u_3^2(s) \end{aligned}$$

To prove part (b) let $\Lambda(s) = X^*(s)\frac{dX}{ds}(s)$. It follows from Lemma 1 that $||\Lambda(s)|| = 1$. Since K acts transitively on S^3 it follows that there exists a curve $R(s)$ such that $R(s)A_1R^*(s) = \Lambda(s)$. Then $(X(s), R(s))$ is the desired anchored Darboux curve. ∎

Remark. The reader may recall that not all curves can be lifted to the Serret-Frenet frames. Darboux frames do not have that disadvantage, as the preceding theorem shows, but the uniqueness is lost. In fact, for any Darboux curve $(X(s), R(s))$ $(X(s), R(s)\phi(s))$ is another anchored Darboux curve that projects onto $X(s)$ provided that $\phi(s)A_1\phi^*(s) = A_1$.

Definition 2.7 *The set of anchored Darboux curves will be denoted by $\mathcal{D}_s(L)$, while $Horiz(\mathcal{D}_s)(L)$ will denote the set of all horizontal Darboux curves.*

Definition 2.8 *An anchored Darboux curve $(X(s), R(s))$ will be called frame- periodic if $R(0) = R(L)$, i.e., if $R(s)$ is a closed curve. The space of frame-periodic Darboux curves will be denoted by $\mathcal{PD}_s(L)$. Then, $Horiz(\mathcal{PD}_s)(L)$ will denote the set of all horizontal frame-periodic Darboux curves.*

If $(X(s), R(s))$ is frame-periodic then both matrices $\Lambda(s)$ and $A(s)$ defined by equations (17) and (18) are smoothly periodic, because closed solutions of differential equations with smooth data are smoothly periodic. However, the projected curve $X(s)$ need not be closed.

On the other hand, the Serret-Frenet frame of any smoothly periodic curve $X(s)$ is periodic, and therefore, all smoothly periodic curves in K are the projections of frame-periodic Darboux curves.

2.3   Darboux curves as Fréchet manifolds

On the basis of the general theory developed in ([6]) both $\mathcal{D}_s(L)$ and $\mathcal{PD}_s(L)$ may be considered as infinite-dimensional Fréchet manifolds. Recall that a topological Hausdorff vector space V is called a Fréchet space if its topology is induced by a countable family of semi-norms p_n, and if it is complete relative to the semi-norms in $\{p_n\}$. A Fréchet manifold is defined as follows:

Definition 2.9 *A Fréchet manifold is a topological Hausdorff space equipped with an atlas whose charts take values in open subsets of a Fréchet space V such that any change of coordinate charts is smooth.*

The paper of R.S. Hamilton ([6]) singles out an important class of Fréchet manifolds, called tame, in which the implicit function theorem is true. One of the main theorems in ([6]) is that the set of smooth mappings from a compact manifold interval into a finite-dimensional Riemannian manifold M is a tame Fréchet manifold. It therefore follows from the implicit function theorem that closed subsets of tame Fréchet manifolds $\mathcal{M}$, defined by the zero sets of finitely many smooth functions on $\mathcal{M}$ are tame sub-manifolds of $\mathcal{M}$. Since both $\mathcal{D}_s(L)$ and $\mathcal{PD}_s(L)$ are particular cases of the above situation, it follows that they are both tame Fréchet manifolds and the same applies to their horizontal projections $Horiz(\mathcal{D}_s)(L)$ and $Horiz(\mathcal{PD}_s)(L)$.

Tangent vectors and tangent bundles of Fréchet manifolds are defined in the same manner as for finite dimensional manifolds. In particular tangent vectors at a point x in a Fréchet manifold $\mathcal{M}$ are the equivalence classes of curves $\sigma(t)$ in $\mathcal{M}$ all emanating from x (i.e.,$\sigma(0) = x$), and all having the same tangent vector $\frac{d\sigma}{dt}(0)$ in each equivalence class. The set of all tangent vectors at x denoted by $T_x\mathcal{M}$ constitutes the tangent space at x.

The tangent bundle of a Fréchet manifold $\mathcal{M}$ is a Fréchet manifold. A vector field X on $\mathcal{M}$ is a smooth mapping from $\mathcal{M}$ into the tangent bundle $T\mathcal{M}$ such that

$X(s) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$. On tame Frécher manifolds vector fields can be defined as derivations in the space of smooth functions on $\mathcal{M}$.

In the formalism of Fréchet manifolds tangent vectors in each of $Horiz(\mathcal{D}_s)(L)$ and $Horiz(\mathcal{PD}_s)(L)$ are given by the following theorem.

Theorem 2 *(a). The tangent space $T_X(Horiz(\mathcal{D}_s)(L)$ at an anchored horizontal Darboux curve $(X(s)$ consists of curves $v(s) = X(s)V(s)$ with $V(s)$ the solution of*

$$\frac{dV}{ds}(s) = [\Lambda(s), V(s)] + U(s) \tag{20}$$

such that $V(0) = 0$, where $U(s)$ is a curve in $\mathfrak{k}$ subject to the conditions that $U(0) = 0$ and $\langle \Lambda(s), U(s) \rangle = 0$. The matrix $\Lambda(s)$ is the tangent vector of X, i.e.,

$$\frac{dX}{ds}(s) = X(s)\Lambda(s)$$

(b). $v(s) = X(s)V(s)$ is a tangent vector at an anchored frame-periodic horizontal Darboux curve $X(s)$ if in addition to the above, the curve $U(s)$ is smoothly periodic having the period equal to L.

Proof. Let $Y(s,t)$ denote a family of anchored horizontal Darboux curves such that $Y(s,0) = X(s)$. Then, $v(s) = \frac{\partial Y}{\partial t}(s,t)_{t=0}$ is a tangent vector at $X(s)$ for which $v(0) = 0$ since the curves $Y(s,t)$ are anchored.

Let $Z(s,t)$ and $W(s,t)$ denote the matrices defined by

$$Z(s,t) = Y(s,t)^* \frac{\partial Y}{\partial s}(s,t), \ W(s,t) = Y(s,t)^* \frac{\partial Y}{\partial t}(s,t)$$

It follows that $\Lambda(s) = Z(s,0)$, $V(s) = W(s,0)$ and that $V(0) = 0$. Furthemore,

$$\frac{\partial Z}{\partial t}(s,t) - \frac{\partial W}{\partial s}(s,t) + [Z(s,t), W(s,t)] = 0$$

as a consequence of Lemma 2. The above equation reduces to

$$\frac{dV}{ds}(s) = [\Lambda(s), V(s)] + U(s)$$

when $t = 0$ provided that $U(s) = \frac{\partial W}{\partial s}(s,0)$.

Since the curves $s \to Y(s,t)$ are Darboux for each t,

$$\langle Z(s,t), Z(s,t) \rangle = 1, \text{ and } Z(0,t) = A_1$$

Therefore,

$$\langle \frac{\partial Z}{\partial t}(s,t), Z(s,t) \rangle = 0, \text{ and } \frac{\partial Z}{\partial t}(0,t) = 0$$

which implies that $\langle \Lambda(s), U(s) \rangle = 0$ and $U(0) = 0$.

It remains to show that any curve $V(s)$ in $\mathfrak{k}$ that satisfies (20) can be realized by the perturbations $Y(s, t)$ used above. So assume that $V(s)$ be any solution of (20) generated by a curve $U(s)$ with $U(0) = 0$ that satisfies $\langle \Lambda(s), U(s) \rangle = 0$.

Let $\phi(t)$ denote any smooth function such that $\phi(0) = 0$ and $\frac{d\phi}{dt}(0) = 1$. Define

$$Z(s, t) = \frac{1}{1 + \phi^2(t)\langle U(s), U(s) \rangle}(\Lambda(s) + \phi(t)U(s))$$

Evidently $Z(0, t) = A_1$ for all t, and an easy calculation shows that $\langle Z(s, t), Z(s, t) \rangle = 1$. Therefore $Y(s, t)$, the solution of

$$\frac{\partial Y}{\partial s}(s, t) = Y(s, t)Z(s, t)$$

with $Y(0, t) = I$ corresponds to an anchored horizontal Darboux curve for each t. Since $U(s) = \frac{\partial Z}{\partial t}(s, 0)$ our proof of part (a) is finished.

To prove part (b) we shall assume that the curves $Y(s, t)$ used in part (a) belong to $Horiz(\mathcal{PD}_s)(L)$. Then, curves $s \to Z(s, t)$ are L-periodic for each t, and therefore, $U(s) = \frac{\partial Z}{\partial t}(s, 0)$ is periodic with period L.

∎

2.4 The symplectic structure of horizontal Darboux curves

The basic notions of symplectic geometry of infinite-dimensional Fréchet manifolds are defined through differential forms in the same manner as for the finite-dimensional situations. In particular, differential forms ω of degree n are mappings

$$\omega : \underbrace{\mathcal{X}(\mathcal{M}) \times \cdots \times \mathcal{X}(\mathcal{M})}_{n} \to C^\infty(\mathcal{M})$$

that are $C^\infty(\mathcal{M})$ multilinear and skew-symmetric. Here $\mathcal{X}(\mathcal{M})$ denotes the space of all smooth vector fields on $\mathcal{M}$.

Definition 2.10 *The exterior derivative $d\omega$ of a form of degree n is a differential form of degree $n + 1$ defined by*

$$
\begin{aligned}
d\omega(X_1, \ldots, X_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1, \ldots, \hat{X}_1, \ldots, X_n)) \\
&\quad - \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], \ldots \hat{X}_i, \ldots \hat{X}_j, X_{n+1})
\end{aligned}
$$

where the roof sign above an entry indicates its absence from the expression (i.e., $w(\hat{X}_1, X_2) = w(X_2)$ and $w(X_1, \hat{X}_2) = w(X_1)$).

A differential form ω is said to be closed if its exterior derivative $d\omega$ is equal to zero.

Definition 2.11 *A differential form ω of degree 2 is said to be symplectic whenever it is closed and non-degenerate, in the sense that the induced form $(i_X\omega)(Y) = \omega(X, Y)$ is non-zero for each non-zero vector field X.*

The differential df of a smooth function f is a form of degree 1 defined by $df(v) = \frac{d}{dt} f \circ \sigma(t)|_{t=0}$ for any smooth curve in $\mathcal{M}$ such that $\sigma(0) = x$, and $\frac{d\sigma}{dt}(0) = v$.

In finite dimensional symplectic manifolds with a symplectic form ω there is a unique vector field X_f such that $df = i_{X_f}\omega$. X_f is called the Hamiltonian vector field induced by f, and f is called the Hamiltonian of X_f. However, in infinite dimensional manifolds it may happen that the form df is not equal to $i_X w$ for any $X \in \mathcal{X}(\mathcal{M})$. This is due to the fact that the cotangent bundle of an infinite dimensional Fréchet space is never a Fréchet manifold. Nevertheless,

Definition 2.12 *A vector field X is said to be Hamiltonian if there exists a smooth function f such that*

$$df(Y) = \omega(X, Y)$$

for all vector fields Y on $\mathcal{M}$. The dependence of X on f shall be noted explicitly by X_f.

The manifold consisisting of horizontal Darboux curves posesses a natural differential form ω defined as follows:

Let $v_1(s) = X(s)V_1(s)$ and $v_2(s) = X(s)V_2(s)$ denote any tangent vectors at a horizontal Darboux curve $X(s)$ that is defined by $\frac{dX}{ds}(s) = X(s)\Lambda(s)$. According to (20) there exist unique curves $U_1(s)$ and $U_2(s)$ such that

$$U_i(0) = 0, \; \langle \Lambda(s), U_i(s) \rangle = 0$$

and

$$U_i(s) = \tfrac{dV_i}{ds}(s) - [\Lambda(s), V_i(s)], \; i = 1, 2. \tag{21}$$

Then ω is given by

$$\omega_\Lambda(V_1, V_2) = -\int_0^L \langle \Lambda(s), [U_1(s), U_2(s)] \rangle \, ds \tag{22}$$

Remark. As in finite dimensional situations the choice of sign is a matter of convention. The justification for the above choice of sign will be given later on in the paper.

Theorem 3 *Both $Horiz(\mathcal{D})_s(L)$ and $Horiz(\mathcal{PD})_s(L)$ are symplectic Fréchet manifolds relative to ω defined by (22).*

The following lemmas will be useful for the proof of the theorem.

Lemma 3
$$[A, [B, C]] = \langle A, C \rangle B - \langle A, B \rangle C)$$

for any elements A, B, C in $\mathfrak{k}$.

We leave the proof to the reader.

Lemma 4 *Suppose that $v(s) = g(s)V(s)$ is any tangent vector at a horizontal Darboux curve $g(s)$. Let $U(s)$ be defined by (20). Then there exists a curve $C(s)$ in $\mathfrak{k}$ such that*
$$U(s) = [\Lambda(s), C(s)]$$

Proof. *The mapping $C \to [\Lambda(s), C(s)]$ restricted to the orthogonal complement of $\Lambda(s)$ is surjective. Since $U(s)$ is orthogonal to $\Lambda(s)$ the proof follows.* ∎

Proof of the theorem. The proof is the same for each of $Horiz(\mathcal{D})_s(L)$ and $Horiz(\mathcal{PD})_s(L)$ and will be presented formally without any reference to the underlying space.

Evidently ω is skew-symmetric. To show that it is non-degenerate, assume that for some tangent vector $v_1 = gV_1$ at g, $\omega_\Lambda(V_1, V) = 0$ for all tangent vectors $v = gV$ at g. Let $U(s)$ and $U_1(s)$ correspond to $V(s)$ and $V_1(s)$ as in (21). Then take $U(s) = [\Lambda(s), U_1(s)]$. Evidently $U(0) = 0$, and $\langle \Lambda(s), U(s) \rangle = 0$, and therefore the corresponding vector $v = gV$ belongs to the tangent space at g. It follows from Lemma 3 that
$$[U_1[\Lambda, U_1] = \langle U_1, U_1 \rangle \Lambda = ||U_1||^2 \Lambda$$

Therefore,
$$\langle \Lambda(s), [U_1(s), U(s)] \rangle = ||\Lambda(s)||^2 ||U_1(s)||^2 = ||U_1(s)||^2$$

which implies that $U_1(s) = 0$ for all s, since $0 = \omega_\Lambda(V_1, V) = \int_0^L ||U_1(s)||^2 \, ds$.

But then (21) implies that $V_1(s) = 0$. Hence, ω is non-degenerate.

To show that ω is closed let $v_i(s) = g(s)V_i(s)$, $1 \le i \le 3$ denote any three tangent vectors at a fixed Darboux curve $g(s)$. It is required to show (Definition 2.10) that

$$d\omega(X_1, X_2, X_3) = \sum_{cyclic} X_i(\omega(X_j, X_k) + \sum_{cyclic} \omega([X_i, X_j], X_k) = 0 \qquad (23)$$

where X_i denote any vector fields such that $X_i(g) = v_i$ for each $i = 1, 2, 3$.

To make use of the above formula it becomes necessary to identify vectors v_i with vector fields X_i. For the purposes of this identification it will be convenient to make a slight change in the notations and write $\Lambda_z(s)$ instead of $\Lambda(s)$ for the tangential direction $\Lambda(s)$ of the horizontal Darboux curve $z(s)$.

Vector fields X_i will be induced by the matrices $U_i = \frac{dV_i}{ds}(s) - [\Lambda, V_i]$ through the following relations: for each horizontal Darboux curve $z(s)$ let $Z_i(s)$ denote the solution of
$$\frac{dZ_i}{ds}(s) = [\Lambda(z)(s), Z_i(s)] + [\Lambda(z)(s), U_i(s)]$$

with $Z_i(0) = 0$ for each $i = 1, 2, 3$. Then

$$X_i(z)(s) = z(s)Z_i(s), \ i = 1, 2, 3$$

To show that (23) is valid let $C_i(s)$ denote the curves such that $U_i(s) = [\Lambda(s), C_i(s)]$ as in Lemma 4. Then, $\frac{dV_i}{ds} = [\Lambda, V_i + C_i]$ and an easy calculation based on Jacobi's identity yields

$$\frac{d}{ds}([V_i, V_j]) = [\Lambda, [V_i, V_j] + [[\Lambda, C_i], V_j] + [V_i, [\Lambda, C_j]]$$

Therefore,

$$
\begin{aligned}
\sum_{cyclic} \omega([X_i, X_j], X_k) &= \sum_{cyclic} \int_0^L \langle \Lambda, [[[\Lambda, C_i], V_j] + [V_i, [\Lambda, C_j]], [\Lambda, C_k]] \rangle \, ds \\
&= \sum_{cyclic} \int_0^L \langle \Lambda, [(\langle V_i, \Lambda \rangle C_j - \langle V_j, \Lambda \rangle C_i) + (\langle V_j, C_i \rangle \Lambda - \langle V_i, C_j \rangle \Lambda), [\Lambda, C_k]] \rangle \, ds \\
&= \sum_{cyclic} \int_0^L \langle V_j, \Lambda \rangle \langle C_i, C_k \rangle - \langle V_i, \Lambda \rangle \langle C_j, C_k \rangle \, ds = 0
\end{aligned}
$$

The calculations involving $X_i(\omega(X_j, X_k)$ in (23) require additional notations. Let $t \to z_i(s, t)$ denote the integral curves of the vector field X_i that originate at $g(s)$ for $t = 0$, and let

$$\frac{\partial z_i}{\partial t}(s, t) = z_i(s, t)Z_i((z_i(s, t)) \text{ and } \frac{\partial z_i}{\partial s}(s, t) = z_i(s, t)\Lambda_i(z_i(s, t)).$$

For simplicity of notation let $Z_i(z_i(s, t))$ and $\Lambda_i(z_i(s, t))$ be denoted by $Z_i(s, t)$ and $\Lambda_i(s, t)$. From Lemma 2

$$\frac{\partial \Lambda_i}{\partial t} - \frac{\partial Z_i}{\partial s} + [\Lambda_i, Z_i] = 0 \tag{24}$$

which at $t = 0$ reduce to

$$U_i - \frac{dV_i}{ds} + [\Lambda, V_i] = 0$$

As in the preceeding calculation U_i will be represented by $U_i = [\Lambda, C_i]$.

Then,

$$
\begin{aligned}
X_i(\omega(X_j, X_k) \;=\;& \frac{\partial}{\partial t} \int_0^L \langle \Lambda_i(s,t), [[\Lambda_j(s,t), C_j], [\Lambda_k(s,t), C_k]] \, ds|_{t=0} \\
=\;& \int_0^L \langle \frac{\partial \Lambda_i}{\partial t}(s,t), [[\Lambda_j(s,t), C_j], [\Lambda_k(s,t), C_k]] \rangle \, ds|_{t=0} \\
+\;& \int_0^L \langle \Lambda_i(s,t), [[\frac{\partial \Lambda_j}{\partial t}(s,t), C_j], [\Lambda_k(s,t), C_k]] \rangle \, ds|_{t=0} \\
+\;& \int_0^L \langle \Lambda_i(s,t), [[\Lambda_j(s,t), C_j], [\frac{\partial \Lambda_k}{\partial t}(s,t), C_k]] \rangle \, ds|_{t=0} \\
=\;& \int_0^L \langle U_i(s), [U_j(s), U_k(s)] \rangle \, ds \\
+\;& \int_0^L \langle \Lambda(s), ([[U_j(s), C_j], [\Lambda(s), C_k]] + [[\Lambda(s), C_j], [U_k(s), C_k]]) \, ds
\end{aligned}
$$

Since $\langle U_i(s), [U_j(s), U_k(s)] \rangle$ is equal to the volume of the parallelopiped with sides $U_i(s), U_j(s), U_k(s)$,

$$
\langle U_i(s), [U_j(s), U_k(s)] \rangle = 0
$$

because each side of the parallelopiped is in the plane orthogonal to $\Lambda(s)$. Therefore,

$$
\int_0^L \langle U_i(s), [U_j(s), U_k(s)] \rangle \, ds = 0
$$

It remains to show that

$$
\sum_{cyclic} \int_0^L \langle \Lambda(s), ([[U_j(s), C_j], [\Lambda(s), C_k]] + [[\Lambda(s), C_j], [U_k(s), C_k]]) \, ds = 0
$$

It follows that

$$
\langle \Lambda(s), ([[U_j(s), C_j], [\Lambda(s), C_k]] + [[\Lambda(s), C_j], [U_k(s), C_k]]) =
$$
$$
\langle \Lambda(s), (\langle [U_j(s), C_j], \Lambda(s) \rangle C_k - \langle [U_j(s), C_j], C_k \rangle \Lambda(s) + \langle [U_j(s), C_j], \Lambda(s) \rangle C_k - \langle [U_k(s), C_k], C_j \rangle \Lambda(s) \rangle) =
$$
$$
\langle [U_k(s), C_k(s)], C_j \rangle - \langle [U_j(s), C_j], C_k \rangle = 0
$$

because $\langle [U_k(s), C_k(s)], C_j \rangle$ is the volume of the parallelopiped with sides $U_k(s), C_k(s), C_j$ all of which are in the plane orthogonal to $\Lambda(s)$. Therefore ω is closed, and hence symplectic.

■

3 The symplectic structure of hyperbolic Darboux curves

3.1 The hyperboloid and its frame bundle

Similar to the sphere, the hyperboloid can be represented in several ways. To make easy transitions from the sphere to the present situation it will be most convenient to represent the hyperboloid as the homogeneous manifold $SL_2(C)/SU_2$. The representation is done as follows.

Each vector in x in $\mathbb{R}^4$, considered as a column vector with coordinates x_0, x_1, x_2, x_3, can be represented by a Hermitian matrix

$$X = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix}$$

. If $(a, b)_h$ denotes the Lorentzian quadratic form in $\mathbb{R}^4$ defined by

$$(a, b)_h = a_0 b_0 - (a_1 b_1 + a_2 b_2 + a_3 b_3) \tag{25}$$

then the Lorentzian unit sphere $(x, x)_h = 1$ is a hyperboloid of two sheets and corresponds to the space of Hermitian matrices X whose determinant is equal to 1.

Definition 3.1 *The connected component of the Lorentzian unit sphere defined by $x_0 > 0$ shall be referred to as the hyperboloid and will be denoted by $\mathbb{H}^3$. As a subset of the Hermitian matrices, the hyperboloid $\mathbb{H}^3$ is equal to the space of positive-definite Hermitian matrices X whose determinant is equal to 1.*

The orthonormal frame bundle of $\mathbb{H}^3$ is equal to $SO(1, 3)$, the matrix group that leaves the Lorentzian form (25) invariant (it is understood here that $SO(1, 3)$ acts on the points of $\mathbb{R}^4$ by the matrix multiplications). The restriction of this action to $\mathbb{H}^3$ is transitive and consequently, $\mathbb{H}^3$ can be identified with the orbit of $SO(1, 3)$ through any of its points $\hat{x}$. In this identifications points of $\mathbb{H}^3$ are represented by the first columns of the matrices $R \in SO(1, 3)$ when $\hat{x} = e_0$. The other columns of the matrix are identified with an orthonormal frame v_1, v_2, v_3 at the base point $x = Re_0$. More precisely, the correspondence between matrices $R \in SO(1, 3)$ and points of the othonormal frame bundle of $\mathbb{H}^3$ is given by the following relations:

$$x = Re_0, \ v_1 = Re_1, \ v_2 = Re_2, \ v_3 = Re_3 \tag{26}$$

The frame e_1, e_2, e_3 at the identity induces an orientation on the space of frames and identifies $SO_0(1, 3)$, the connected component of $SO(1, 3)$ that contains the identity, as the positively oriented orthonormal frame bundle of $\mathbb{H}^3$.

The subgroup H of $SO_0(1, 3)$ that fixes e_0 is isomorphic to $SO_3(R)$ and acts by the right multiplications on each fiber of the projection map $\pi(R) = Re_0$. In the

language of the principal bundles the above construction identifies $SO(1,3)$ as the principal H bundle with $\mathbb{H}^3$ the base space under the projection map π.

The transition from the spherical case to the hyperbolic case is more direct if instead of the realization $SO(1,3)/SO_3(R)$ the hyperboloid is realized as the quotient $SL_2(C)/SU_2$. The passage from $SO(1,3)$ to $SL_2(C)$ is obtained through the identification of points $x = (x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$ with Hermitian matrices

$$X = \left(\begin{array}{cc} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{array} \right)$$

and a homomorphism $\Phi(p) = R$ from $SL_2(C)$ onto $SO(1,3)$ defined by :

$$pXp^* = Y \text{ if and only if } Rx = y, \ x \in R^4 \tag{27}$$

That the matrix R defined by (27) is in $SO(1,3)$ can be easily seen through the following lemma.

Lemma 5 *Suppose X and Y are Hermitian matrices that correspond to points x and y in $\mathbb{R}^4$. Then,*
(a). $\frac{1}{2}(XY^\dagger + YX^\dagger) = (x,y)_h I$.
(b). $(gX)(gY)^\dagger + (gY)(gX)^\dagger = XY^\dagger + YX^\dagger$ for any g in $SL_2(C)$.

In the statement of the lemma $X^\dagger$ denotes the companion matrix of a Hermitian matrix X. Recall that

$$X^\dagger = \left(\begin{array}{cc} x_0 - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_0 + x_1 \end{array} \right) \text{ for any } X = \left(\begin{array}{cc} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{array} \right).$$

The proof of this lemma will be left to the reader.

The homomorphism Φ given by (27) shows that $SL_2(C)$ is a double cover of $SO(1,3)$ since $ker(\Phi) = \{\pm I\}$. Consequently, the restriction of Φ to $H = \{1\} \times SO_3(R)$ proves SU_2 is a double cover of $SO_3(R)$. Additionally, the action $(g, X) \rightarrow gXg^*$ identifies $\mathbb{H}^3$ as the quotient $SL_2(C)/SU_2$ via the orbit through the identity.

The frame $v_1 = Re_1, v_2 = Re_2, v_3 = Re_3$ at $x = Re_0$ corresponds to the matrices

$$V_1 = pE_1p^*, \ V_2 = pE_2p^*, \ V_3 = pE_3p^* \tag{28}$$

where

$$E_0 = I, E_1 = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), E_2 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), E_3 = \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right)$$

are the Hermitian matrices that correspond to the standard basis e_0, e_1, e_2, e_3 of $\mathbb{R}^4$.

3.2   Lie algebras and the trace form

The Lie algebra of $SL_2(C)$ consisting of 2×2 complex matrices of trace zero shall be denoted by $\mathfrak{g}$. The vector space of all Hermitian matrices in $\mathfrak{g}$ will be denoted by

$\mathfrak{h}$ and the Lie subalgebra of all skew-Hermitian matrices in $\mathfrak{g}$ will be denoted by $\mathfrak{k}$ in accordance with the notations used earlier in the paper. Then,

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$$

and the following Lie algebraic relations hold

$$[\mathfrak{h}, \mathfrak{h}] = \kappa, \quad [\mathfrak{h}, \mathfrak{k}] = \mathfrak{h} \quad \text{and} \quad [\kappa, \mathfrak{k}] = \mathfrak{k} \tag{29}$$

The relation $[\mathfrak{h}, \mathfrak{k}] = \mathfrak{h}$ easily implies that $\varphi A \varphi^{-1} \in \mathfrak{h}$ for any $\varphi \in K$ and any A in $\mathfrak{h}$, which further implies that the same holds for absolutely continuous curves

$$\varphi(s) A(s) \varphi^{-1}(s) \tag{30}$$

with $\varphi(s)$ a curve in K and $A(s)$ a curve in $\mathfrak{h}$.

Definition 3.2 *The matrices*

$$B_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, B_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

shall be called the Hermitian Pauli matrices.

The Hermitian Pauli matrices are related to the skew-Hermitian matrices A_1, A_2, A_3 defined earlier by the following simple correspondences

$$A_1 = iB_1, A_2 = iB_2, A_3 = iB_3 \tag{31}$$

Matrices $B_1, B_2, B_3, A_1, A_2, A_3$ form a basis for $\mathfrak{g}$, and the reader can readily verify that their Lie brackets conform to the following Lie bracket table:

$[\ ,\ ]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	A_3	$-A_2$
B_2	B_3	0	$-B_1$	$-A_3$	0	A_1
B_3	$-B_2$	B_1	0	A_2	$-A_1$	0

Table 1

It may be helpful for some of the subsequent calculations to note the following relations:

Lemma 6 *(a) If A and B are skew-Hermitian with $A = \sum_{i=1}^{3} a_i A_i$ and $B = \sum_{i=1}^{3} b_i A_i$, then $[A, B] = \sum_{i=1}^{3} c_i A_i$ where c is the vector product $b \times a$ in R^3.*

(b) If A is skew-Hermitian with $A = \sum_{i=1}^{3} a_i A_i$, and B is Hermitian with $B = \sum_{i=1}^{3} b_i B_i$, then again $[A, B] = \sum_{i=1}^{3} c_i B_i$ with $c = b \times a$.

(c) However, if both A and B are Hermitian with $A = \sum_{i=1}^{3} a_i B_i$ and $B = \sum_{i=1}^{3} b_i B_i$ then $[A, B] = \sum_{i=1}^{3} c_i A_i$ with $c = a \times b$. Thus for Hermitian matrices the order in the cross product is reversed.

Definition 3.3 *For any matrices A, B in $\mathfrak{g}$, $\langle A, B \rangle$ will denote the trace of $2(AB)$. We shall refer to this quadratic form as the trace form on $sl_2(C)$.*

The Hermitian Pauli matrices B_1, B_2, B_3 form an orthonormal basis in $\mathfrak{h}$ relative to the trace form, which together with (31) implies that

$$\langle A_i, A_j \rangle = -\delta_{ij} \text{ and } \langle A_i, B_j \rangle = i\delta_{ij}$$

Remark. The above relations imply that the restriction of the trace form to the matrices in $\mathfrak{k}$ is the negative of the trace form used in the first part of the paper. For the most part of the paper this ambiguity in terminology will not matter since the meaning will be clear from the context, but in the instances that require clarification the meaning will be made explicit.

It follows from above that

$$\langle A, A \rangle = -(a_1^2 + a_2^2 + a_3^2), \langle B, B \rangle = b_1^2 + b_2^2 + b_3^2, \text{ and } \langle A, B \rangle = i(a_1 b_1 + a_2 b_3 + a_3 b_2)$$

for any matrices $A = \sum a_j A_j$ and $B = \sum b_j B_j$. The above imply that

$$\mathfrak{h} \cong \mathfrak{k}^*$$

where $\mathfrak{k}^*$ denotes the dual of $\mathfrak{k}$, a fact of central importance for the results that follow.

3.3 Hyperbolic Darboux curves

As in the first part of the paper horizontal Darboux curves shall are naturally introduced through the language of the principal bundles. Let G denote the group $SL_2(C)$ and let π denote the projection map from G onto the base manifold $\mathbb{H}^3$ given by $\pi(g) = X = gg^*$. Then π_* will denote the tangent map of π. It follows that $\pi_*(gA) = gAg^*$ for every left invariant vector field $g \to gA$ in G. The group K acts on G by right translations and the action is invariant and transitive on each fiber $\pi^{-1}(g)$. Therefore $(G, \mathbb{H}^3, \pi, K)$ is a principal K-bundle.
.

Definition 3.4 *The distribution spanned by the left-invariant vector fields $V(g) = gA$ with A in $\mathfrak{h}$ and $g \in G$ shall be called horizontal and will be denoted by $\mathcal{H}$. The integral curves of $\mathcal{H}$ in G shall be called horizontal curves. Vertical curves are the integral curves of the left invariant distribution with values in $\mathfrak{k}$. A horizontal curve $g(s)$ is called a horizontal lift of a curve $X(s)$ in $\mathbb{H}^3$ if $\pi(g(s)) = X(s)$ for all s.*

Theorem 4 *(a). Every curve $X(s)$ in $\mathbb{H}^3$ can be lifted to a horizontal curve $g(s)$. For any horizontal lifts $g_1(s)$ and $g_2(s)$ of $X(s)$ there exists a vertical curve $a(s)$ such that $g_2(s) = g_1(s)a(s)$.*

(b). If a horizontal lift $g(s)$ of a base curve $X(s)$ is given by $\frac{dg}{ds}(s) = g(s)B(s)$ and $B(s) = \sum b_i(s)B_i$, then

$$||\frac{dX}{ds}(s)||^2 = 4\langle B(s), B(s)\rangle = 4(b_1^2(s) + b_2^2(s) + b_3^2(s)).$$

where

$$||\frac{dX}{ds}||^2 = -\frac{dx_0}{ds}^2 + \frac{dx_1}{ds}^2 + \frac{dx_2}{ds}^2 + \frac{dx_3}{ds}^2 = -(\frac{dx}{ds}, \frac{dx}{ds})_h$$

Proof. Let $g(s)$ denote any curve in G that projects onto a given curve $X(s)$ in $\mathbb{H}^3$ and let $A(s)$ and $B(s)$ denote curves in $\mathfrak{g}$ with $A(s) \in \mathfrak{k}$ and $B(s) \in \mathfrak{h}$ such that

$$\frac{dg}{ds}(s) = g(s)(B(s) + A(s)).$$

Let $g_0(s) = g(s)\phi^{-1}(s)$ where $\phi(s)$ denote any solution of

$$\frac{d\phi}{ds}(s) = \phi(s)A(s).$$

Then, $g_0(s)$ projects onto $X(s)$ because $\phi(s)$ is a vertical curve, and furthemore,

$$\frac{dg_0}{ds}(s) = g_0(s)(\phi(s)B(s)\phi^{-1}(s)).$$

The relations (29) imply that $\phi(s)B(s)\phi^{-1}(s)$ belongs to $\mathfrak{h}$, and therefore $g_0(s)$ is a horizontal curve. This proves the first statement in (a). The second statement in (a) follows directly from the definition of a principal bundle.

To prove part (b), let $g(s)$ ba a horizontal curve that projects onto $X(s)$ such that

$$\frac{dg}{ds}(s) = g(s)B(s)$$

Since $X(s) = g(s)g^*(s)$, $\frac{dX}{ds}(s) = 2g(s)B(s)g^*(s)$.

Then Lemma 5 implies that

$$
\begin{aligned}
(||\frac{dX}{ds}(s)||^2))I &= -(\frac{dX}{ds}(s))(\frac{dX}{ds}(s))^\dagger = -4(g(s)B(s)g^*(s))(g(s)B(s)g^*(s)^\dagger) \\
&= -4(g(s)B(s)g^*(s)(g^*(s)^\dagger B^\dagger(s)g^\dagger(s) = -4B(s)B^\dagger(s) \\
&= 4(\langle B(s), B(s)\rangle)I
\end{aligned}
$$

Therefore,

$$\|\frac{dX}{ds}(s)\|^2) = 4\langle B(s), B(s)\rangle)$$

∎

Definition 3.5 *The hyperboloid* $\mathbb{H}^3$ *shall be considered a Riemannian manifold with its metric given by* $\frac{1}{2}\sqrt{\frac{dx_0}{ds}^2 + \frac{dx_1}{ds}^2 + \frac{dx_2}{ds}^2 + \frac{dx_3}{ds}^2}$.
Then previous lemma implies

Corollary 1 *Horizontal curves* $g(s)$ *defined by* $\frac{dg}{ds}(s) = g(s)B(s)$ *project onto curves* $X(s)$ *parametrized by arc-length, i.e.* $\|\frac{dX}{ds}(s)\| = 1$, *if and only if* $\langle B(s), B(s)\rangle = 1$.

For the remainder of the paper all curves $g(s)$ in G will be anchored at the identity $(g(0) = I)$, and due to the relation (27) will be identified with the curves in the orthonormal frame bundle of $\mathbb{H}^3$. As in the spherical case

Definition 3.6 *Curves* $g(s)$ *in* G *will be called Darboux if* $v_1(s) = g(s)B_1g^*(s)$ *is equal to the tangent vector* $\frac{dX}{ds}(s)$ *of the projected curve* $X(s) = \pi(g(s))$.

The above condition means that Darboux curves are the solutions of

$$\frac{dg}{ds}(s) = g(s)(B_1 + A(s))$$

for arbitrary curves $A(s)$ in $\mathfrak{k}$.

Definition 3.7 *Horizontal curves* $g(s)$ *will be called horizontal anchored Darboux curves, or simply horizontal Darboux, if* $g(0) = I$ *and*

$$g^{-1}(s)\frac{dg}{ds}(s) = \Lambda(s) = R(s)B_1R^*(s)$$

for some curve $R(s)$ *in* K *with* $R(0) = I$.

Since K acts transitively on the sphere S^3, the horizontal Darboux curves could have been defined equivalently as the solutions $g(s)$ of the differential equation

$$\frac{dg}{ds}(s) = g(s)\Lambda(s)$$

such that $g(0) = I$, subject to further conditions that $\Lambda(s) \in \mathfrak{h}$, $\Lambda(0) = 0$, and $\langle \Lambda(s), \Lambda(s)\rangle = 1$ for all $s \in [0, L]$.

Definition 3.8 *The space of all hyperbolic Darboux curves and horizontal Darboux curves shall be denoted by* $\mathcal{D}_h(L)$ *and* $Horiz(\mathcal{D}_h)(L)$.

Definition 3.9 *The covariant derivative* $\frac{D_g}{ds}(v)$ *of a curve of tangent vectors* $g(s)V(s)$, $V(s) \in \mathfrak{b}$, *along a horizontal curve* $g(s)$ *in* G, *is defined by* $\frac{D_g}{ds}(gV)(s) = g(s)\frac{dV}{ds}(s)$ *for all* $s \in [0, L]$.

This notion of covariant derivative for vectors in the horizontal distribution $\mathcal{H}$ coincides with the usual notion of covariant derivative in the base manifold $\mathbb{H}^3$ in the sense that

$$\frac{D_{\pi(g)}}{ds}(\pi_*(gV))(s) = \pi_*((g(s)\frac{dV}{ds}))(s)$$

The proof is simple and goes as follows:

The covariant derivative in $\mathbb{H}^3$ inherited from the ambient manifold $\mathbb{R}^4$ is analogous to the spherical case, except that the Euclidean inner product in $\mathbb{R}^4$ is replaced by the Lorentzian inner product, and is given by the following formula:

$$\frac{D_x}{ds}(v)(s) = \frac{dv}{ds}(s) + \frac{1}{4}(\frac{dx}{ds}(s), v(s))_h x(s)$$

Remark The factor $\frac{1}{4}$ appears because the metric in this paper is equal to the half of the standard hyperbolic metric.

To adapt this formula to the representation by Hermitian matrices, first note that if V and Λ are any Hermitian matrices in $\mathfrak{g}$ then

$$V^\dagger = -V \text{ and } \Lambda^\dagger = -\Lambda.$$

Then if $\pi_*(g(s)V(s))$ is a curve of tangent vectors defined along a curve $X(s) = \pi(g(s)) = g(s)g^*(s)$ corresponding to a horizontal curve $g(s)$ in G that is a solution of

$$\frac{dg}{ds}(s) = g(s)\Lambda(s),$$

then the projected vectors v and λ are given by $v = \pi_*(gV) = 2gVg^*$ and $\lambda = \pi_*(g\Lambda) = 2g\Lambda g^*$. The results of Lemma 5 show that

$$(v, \lambda)_h I = -\langle \Lambda, V \rangle I$$

It then follows that

$$
\begin{aligned}
\frac{D_X}{ds}(\pi_*(g(s)V(s)))(s) &= 2\frac{d}{ds}(g(s)V(s)g^*)(s) - 4\langle V, \Lambda \rangle (g(s)g^*(s)) \\
&= 2g(s)(\Lambda(s)V(s) + V(s)\Lambda(s))g^*(s) + 2g(s)\frac{dV}{ds}(s)g^*(s) - (\langle V, \Lambda \rangle)g(s)g^*(s) \\
&= 4\langle V, \Lambda \rangle)g(s)g^*(s) + 2g(s)\frac{dV}{ds}(s)g^*(s) - 4\langle V, \Lambda \rangle)g(s)g^*(s) \\
&= 2g(s)\frac{dV}{ds}(s)g^*(s)
\end{aligned}
$$

$\blacksquare$

Theorem 5 *Suppose that $g(s,t)$ is a field of horizontal curves in G. Let $V(s,t) = g(s,t)^{-1}\frac{\partial g}{\partial t})(s,t)$ and $\Lambda(s,t) = g(s,t)^{-1}\frac{\partial g}{\partial s})(s,t)$. Then,*

$$\frac{\partial \Lambda}{\partial t}(s,t) = \frac{\partial V}{\partial s}(s,t)$$

and

$$\frac{D_g}{ds}(\frac{\partial g}{\partial t})(s,t) = \frac{D_g}{dt}(\frac{\partial g}{\partial s})(s,t)$$

Proof. Let $\pi(g(s,t)) = X(s,t)$. Then, as in any Riemannian manifold,

$$\frac{D_X}{ds}(\frac{\partial X}{\partial t})(s,t) = \frac{D_X}{dt}(\frac{\partial X}{\partial s})(s,t)$$

The calculation that preceeds the statement of the theorem shows that

$$\frac{D_X}{ds}(\frac{\partial X}{\partial t})(s,t) = 2g(s,t)\frac{\partial V}{\partial s}(s,t)g^*(s,t) \text{ and } \frac{D_X}{dt}(\frac{\partial X}{\partial s})(s,t) = 2g(s,t)\frac{\partial \Lambda}{\partial t}(s,t)g^*(s,t)$$

Hence,

$$2g(s,t)\frac{\partial V}{\partial s}(s,t)g^*(s,t) = 2g(s,t)\frac{\partial \Lambda}{\partial t}(s,t)g^*(s,t)$$

and consequently,

$$\frac{\partial V}{\partial s}(s,t) = \frac{\partial \Lambda}{\partial t}(s,t)$$

The remaining statement follows directly from Definition 3.9 ∎

Theorem 6 *The curvature $\kappa(s)$ of the projection curve $X(s) = \pi(g(s))$ that corresponds to a horizontal curve $g(s)$ conforms to the following equation*

$$||\kappa(s)||^2 = \langle \frac{d\Lambda}{ds}, \frac{d\Lambda}{ds} \rangle$$

where $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$.
Proof.

$$\begin{aligned} ||\kappa(s)||^2 &= ||\frac{D_X}{ds}((\frac{dX}{ds})(s)||^2 \\ &= ||\pi_*(\frac{D_g}{ds}(\frac{dg}{ds}))||^2 = ||\pi_*(g(s)\frac{d\Lambda}{ds})||^2 \\ &= \langle \frac{d\Lambda}{ds}, \frac{d\Lambda}{ds} \rangle \end{aligned}$$

∎

When a horizontal curve $g(s)$ is a Darboux curve then its tangent vector $\Lambda(s)$ is of the form

$$\Lambda(s) = R(s)B_1R^*(s)$$

for some curve $R(s)$ in K that originates at the identity for $s = 0$. If $R(s)$ is a solution of

$$\frac{dR}{ds}(s) = R(s)A(s) = \sum u_i(s)A_i, then$$

$$\frac{d\Lambda}{ds}(s) = R(s)[B_1, A(s)]R^*(s) = R(s)(u_3(s)B_2 - u_2(s)B_3)R^*(s)$$

, and therefore

$$||\kappa(s)||^2 = u_2^2(s) + u_3^2(s).$$

The Serret-Frenet frame of the projected curve $X(s) = \pi(g(s))$, defined by the tangent vector $T(s)$, the normal vector $N(s)$ and the binormal vector $B(s)$ evolves according to the well known equations:

$$\frac{D_X}{ds}(T)(s) = \kappa(s)N(s), \frac{D_X}{ds}(N)(s) = -\kappa(s)T(s) + \tau(s)B(s), \frac{D_X}{ds}(B)(s) = -\tau(s)N(s).$$

When represented by the equations (28) the Serret-Frenet vectors take the following form:

$$
\begin{aligned}
T(s) &= \frac{1}{2}R(s)E_1R^*(s) = R(s)B_1R^*(s) \\
N(s) &= \frac{1}{2}R(s)E_2R^*(s) = R(s)B_3R^*(s) \\
B(s) &= \frac{1}{2}R(s)E_3R^*(s) = -R(s)B_2R^*(s).
\end{aligned}
$$

An easy calculation shows that

$$u_1(s) = \tau(s), \ u_2(s) = -\kappa(s), \ u_3(s) = 0. \tag{32}$$

3.4   The symplectic form

Similar to the spherical case, the space of anchored hyperbolic horizontal Darboux curves will be considered an infinite dimensional Fréchet manifold.

Definition 3.10 *An anchored horizontal Darboux curve $g(s)$ will be called frame-periodic if the tangent vector $\Lambda(s)$ is periodic with period L. The set of frame-periodic anchored horizontal Darboux curves will be denoted by $Horiz(\mathcal{PD}_h)(L)$.*

Frame-periodicity requires periodicity of the curve $R(s)$ which in turn, imposes periodicity of its tangent vector $A(s) = \sum U_i(s)A_i$. The projections $X(s)$ in $\mathbb{H}^3$ of frame periodic Darboux curves necessarily have periodic curvature and torsion, but need not be closed.

On the other hand, smoothly closed curves $X(s)$ in $\mathbb{H}^3$ generate periodic Serret-Frenet frames, and therefore can be lifted always to periodic Darboux curves $g(s)$. If $g(s)$ is the periodic solution of

$$\frac{dg}{ds}(s) = g(s)(B_1 + \sum u_i(s)A_i)$$

with g(0)=I, then the corresponding horizontal curve $h(s) = g(s)R^*(s)$ is periodic if and only if $R(s)$ is a periodic solution of

$$\frac{dR}{ds}(s) = R(s)(\sum u_i(s)A_i).$$

For curves on the sphere and in a Euclidean space $R(s)$ can be identified with an orthonormal frame along the curve, which implies that smoothly periodic curves can be always lifted to periodic horizontal Darboux curves. For curves on the hyperboloid however, such a statement requires a proof, since $R(s)$ does not necessarily coincide with an orthonormal frame along the curve.

Theorem 7 *(a) The tangent space $T_g(Horiz(\mathcal{D}_g h)(L)$ at an anchored horizontal Darboux curve $(g(s)$ consists of tangent curves $v(s) = g(s)V(s)$ with $V(s)$ a Hermitian curve that satisfies the following conditions:*

$$V(0) = 0, \ \frac{dV}{ds}(0) = 0, \ \langle \Lambda(s), \frac{dV}{ds}(s)\rangle = 0 \tag{33}$$

where,

$$\frac{dg}{ds}(s) = g(s)\Lambda(s).$$

(b) For anchored frame-periodic horizontal Darboux curves $v(s) = g(s)V(s)$ is a tangent vector at $g(s)$ if in addition to the properties in (a) the curve $\frac{dV}{ds}(s)$ is smoothly periodic having the period equal to L.

Proof. Let $h(s,t)$ denote a family of anchored horizontal Darboux curves such that $h(s,0) = g(s)$. Then $v(s) = \frac{\partial h}{\partial t}(s,t)_{t=0}$ is a tangent vector at $g(s)$. Since the curves $h(s,t)$ are anchored, $v(0) = 0$.

Let $Z(s,t)$ and $W(s,t)$ denote the matrices defined by

$$Z(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial s}(s,t), \ W(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial t}(s,t).$$

It follows that $\Lambda(s) = Z(s,0)$ and $v(s) = g(s)V(s)$ with $V(s) = W(s,0)$. Then $v(0) = 0$ implies that $V(0) = 0$. Furthemore,

$$\frac{\partial Z}{\partial t}(s,t) = \frac{\partial W}{\partial s}(s,t)$$

as a consequence of Lemma 5. For $t = 0$ the above equation reduces to

$$\frac{dV}{ds}(s) = \frac{\partial W}{\partial s}(s,0) = U(s).$$

Since the curves $s \to Y(s,t)$ are Darboux for each t,

$$\langle Z(s,t), Z(s,t)\rangle = 1, \text{ and } Z(0,t) = B_1$$

Therefore,

$$\langle \frac{\partial Z}{\partial t}(s,t), Z(s,t) \rangle = 0, \text{ and } \frac{\partial Z}{\partial t}(0,t) = 0$$

which implies that $\langle \Lambda(s), U(s) \rangle = 0$ and $U(0) = 0$.

Conversely any curve $V(s)$ in $\mathfrak{h}$ that satisfies (33) can be realized by the perturbations $h(s,t)$ defined in the first part of the proof. The argument is similar to the one used for the spherical Darboux curves and goes as follows.

Let $V(s)$ be a curve specified by (33), and let $\frac{dV}{ds}(s) = U(s)$.

Define

$$Z(s,t) = \frac{1}{1 + \phi^2(t)\langle U(s), U(s) \rangle}(\Lambda(s) + \phi(t)U(s))$$

where $\phi(t)$ denote any smooth function such that $\phi(0) = 0$ and $\frac{d\phi}{dt}(0) = 1$. Evidently, $Z(0,t) = B_1$ for all t, and an easy calculation shows that $\langle Z(s,t), Z(s,t) \rangle = 1$. Therefore the solution of

$$\frac{\partial h}{\partial s}(s,t) = h(s,t)Z(s,t)$$

with $h(0,t) = I$ corresponds to an anchored horizontal Darboux curve for each t. Since $U(s) = \frac{\partial Z}{\partial t}(s,0)$ the proof of part (a) is finished.

To prove part (b) assume that the curves $h(s,t)$ used in part (a) belong to $Horiz(\mathcal{PD}_s)(L)$. Then curves $s \to Z(s,t)$ are periodic with period L for each t. Since curves $s \to \frac{\partial Z}{\partial t}(s,t)(s,t)$ preserve this periodicity, $U(s) = \frac{\partial Z}{\partial t}(s,0)$ is periodic with period L.

∎

Definition 3.11 *The symplectic form ω on the space of anchored horizontal hyperbolic Darboux curves is defined as follows*

$$\omega_{g\Lambda}(V_1, V_2) = \frac{1}{i} \int_0^L \langle \Lambda(s), [\frac{DV_1}{ds}(s), \frac{dV_2}{ds}(s)] \rangle \, ds \tag{34}$$

for any tangent vectors $V_2(s), V_2(s)$ at a horizontal curve $g(s)$.

Remark As it stands the above definition may be problematic, since the issues of non-degeneracy and closedness of ω have not been dealt with yet. However, such questions are easily removed because the hyperbolic form ω is isomorphic to its spherical analogue. The proof is as follows:

Let

$$\tilde{\Lambda} = i\Lambda, \; \tilde{U}_1 = i\frac{dV_1}{ds}, \; \tilde{U}_2 = i\frac{dV_2}{ds}.$$

It follows from (29) and Table 1 that $\tilde{\Lambda}, \tilde{U}_1, \tilde{U}_2$ all belong to $\mathfrak{k}$ and that $\langle \tilde{\Lambda}, \tilde{U}_i \rangle = 0$ for $i = 1, 2$. Therefore,

$$\begin{aligned}
\omega_\Lambda(V_1, V_2) &= \frac{1}{i} \int_0^L \langle \Lambda(s), [\frac{DV_1}{ds}(s), \frac{dV_2}{ds}(s)] \rangle \, ds \\
&= -\int_0^L \langle \tilde{\Lambda}(s), [\tilde{U}_1(s), \tilde{U}_2(s)] \rangle \, ds = \tilde{\omega}_{\tilde{\Lambda}}(\tilde{V}_1, \tilde{V}_2)
\end{aligned}$$

under the identification of the curve $X(s)$ in K with the solution of

$$\frac{dX}{ds}(s) = X(s)\tilde{\Lambda}(s)$$

that satisfies $X(0) = I$.

It follows that $\tilde{\omega}$ coincides with the symplectic form for the spherical horizontal Darboux curves. The isomorphism, apart from justifying the choice of sign in (22) also makes transparent the proof of

Theorem 8 *Both $Horiz(\mathcal{D}_h)(L)$ and $Horiz(\mathcal{PD}_h)(L)$ are symplectic manifolds relative to ω defined by (3.11).*

Remark 1 *It may be appropriate to point out that both the spherical and the hyperolic symplectic form in this paper may be seen as natural adaptations to curves of the standard symplectic form ω on the sphere S^2 given explicitly by*

$$\omega_\gamma(a, b) = \gamma \cdot (a \times b)$$

where a an b are tangent vectors at a point γ on S^2. In the light of this observation, the symplectic structure of anchored curves with periodic frames is isomorphic to the symplectic structure of anchored loops on the sphere.

In each of the two cases the right action of SU_2 extends to the space of horizontal anchored Darboux curves with $(g, a) \rightarrow ag(s)a^*$ for each horizontal curve $g(s)$ and each $a \in SU_2$. This action is symplectic relative to the forms used in this paper and $J(g) = \int_0^L \Lambda(s)\,ds$ is the moment map associated with this action (under the implicit assumption that the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is identified with $\mathfrak{g}$ via the trace form).

The moment map induces a function $J_A(g) = \int_0^L \langle \Lambda(s), A \rangle\,ds$ on the space of horizontal anchored curves for each element $A \in \mathfrak{g}$. The Hamiltonian vector field induced by this function coincides with the infinitesimal generator of the action-induced one-parameter group of transformations $\{e^{tA}\gamma e^{-tA}\}$. Then it is well known ([1]) that J is an integral of motion for each Hamiltonian function which is invariant under the action.

The moment map will be taken up again in the problems of mathematical physics further down in the text. There is another symplectic form on the space of anchored curves given by the following expression:

$$\Omega_{g\Lambda}(V_1, V_2) = \int_0^L \langle \Lambda, [V_1, V_2] \rangle\,ds$$

Such a form is mentioned elsewhere in the literature (see for instance [2],[3], and [13]). This form is compatible in the sense of Magri ([15]) with the form used in this paper and can be used to get the integrability results for systems which are bi-Hamiltonian (as outlined in ([2]). However, such investigations seem too particular for the scope of this paper and will not be pursued here.

4 The Hamiltonian flow of $\frac{1}{2}\int_0^L k^2(s)\,ds$

Horizontal Darboux curves $g(s)$ are parametrized by the matrices $\Lambda(s) = R(s)E_1R^*(s)$ with curves $R(s)$ the solutions of

$$\frac{dR}{ds}(s) = R(s)\sum u_i(s)A_i$$

in K with $R(0) = I$. The matrix E_1 is equal to A_1 in the spherical case and equal to B_1 in the hyperbolic case.

Theorems 1 and 6 show that $||\frac{d\Lambda}{ds}(s)||^2 = \kappa^2(s)$ where $\kappa(s)$ is the curvature of the projected curve in the base manifold. Hence the function

$$f(g) = \frac{1}{2}\int_0^L ||\frac{d\Lambda}{ds}(s)||^2\,ds$$

on the space of anchored Darboux curves may be also seen as the function

$$f(x) = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$$

on the space of projected curves $x(s) = \pi(g(s))$.

As a function on the space of horizontal Darboux curves f defines a Hamiltonian vector field $\mathcal{X}_f$ through the symplectic formalism described in the first part of the paper. Remarkably, $\mathcal{X}_f$ leads to Heisenberg's magnetic equation and the non-linear Schroedinger's equation when f is restricted to the frame-periodic Darboux curves. A derivation of this fact, together with the connections to the known results in the literature constitute the subject matter for the remaining part of the paper.

4.1 Heisenberg's magnetic equation

Although conceptually alike, the calculations in the spherical setting are different in several aspects from those in the hyperbolic setting and will be done separately in each of the above mentioned cases.

Hyperbolic Darboux curves. To calculate the directional derivative $df_\Lambda(V)$, let $\hat{g}(s,t)$ be a family of anchored horizontal Darboux curves that are the solutions of

$$\frac{\partial\hat{g}}{\partial s} = \hat{g}(s,t)\hat{\Lambda}(s,t)$$

such that

$$\hat{g}(s,0) = g(s),\ \hat{\Lambda}(s,0) = \Lambda(s),\ \frac{\partial\hat{\Lambda}}{\partial t}(0,s) = \frac{dV}{ds}(s)$$

.

The directional derivative $df_\Lambda(V)$ is given by

$$df_\Lambda(V) = \frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial\hat\Lambda}{\partial s}(s,t), \frac{\partial\hat\Lambda}{\partial s}(s,t) \right\rangle ds|_{t=0}.$$

It follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial\hat\Lambda}{\partial s}(s,t), \frac{\partial\hat\Lambda}{\partial s}(s,t) \right\rangle ds|_{t=0} = \int_0^L \left\langle \frac{\partial\hat\Lambda}{\partial s}(s,t), \frac{\partial}{\partial s}\frac{\partial\hat\Lambda}{\partial t}(s,t) \right\rangle ds|_{t=0}$$

$$= \int_0^L \left\langle \frac{d\Lambda}{ds}, \frac{d}{ds}\left(\frac{dV}{ds}\right) \right\rangle ds$$

$$= -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dV}{ds} \right\rangle ds + \left\langle \frac{d\Lambda}{ds}, \frac{dV}{ds} \right\rangle |_{s=0}^{s=L}.$$

In the space of frame-periodic horizontal Darboux curves the boundary terms $\left\langle \frac{d\Lambda}{ds}, \frac{dV}{ds} \right\rangle |_{s=0}^{s=L}$ are equal to 0 because of periodicity. Consequently,

$$df_\Lambda(V) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dV}{ds} \right\rangle ds.$$

The Hamiltonian vector field is of the form $\mathcal{X}_f(g) = gF$ for some Hermitian matrix $F(s)$ that satisfies

$$df_\Lambda(V) = \frac{1}{i}\int_0^L \left\langle \Lambda(s), [\frac{dF}{ds}, \frac{dV}{ds}] \right\rangle ds$$

for an arbitrary tangential direction $V(s)$. The above is equivalent to

$$\int_0^L \left\langle (\frac{d^2\Lambda}{ds^2} + \frac{1}{i}[\Lambda(s), \frac{dF}{ds}]), \frac{dV}{ds} \right\rangle ds = 0. \tag{35}$$

It then follows from Theorem 7 that $V(s)$ can be taken as $V(s) = i\int_0^s [\Lambda(t), U(s)]\,dt$ where $U(s)$ is an arbitrary curve of Hermitian matrices with $U(0) = 0$. Then equation (35) becomes

$$\int_0^L \left\langle (i[\frac{d^2\Lambda}{ds^2}, \Lambda(s)] + [[\Lambda(s), \frac{dF}{ds}], \Lambda(s)]), U(s) \right\rangle ds = 0.$$

It follows that

$$i[\tfrac{d^2\Lambda}{ds^2}, \Lambda(s)] + [[\Lambda(s), \tfrac{dF}{ds}], \Lambda(s)] = 0 \qquad (36)$$

because $U(s)$ is sufficiently arbitrary. The following Lie bracket identity is true

$$[[A, B], C] = \langle B, C \rangle A - \langle A, C \rangle B$$

for any Hermitian matrices A, B, C as can be readily verified through Table 1.

Therefore,

$$[\Lambda(s), [\Lambda(s), \frac{dF}{ds}]] = -\frac{dF}{ds}$$

and equation (36) becomes

$$i[\frac{d^2\Lambda}{ds^2}, \Lambda(s)] - \frac{dF}{ds} = 0$$

It follows that

$$\mathcal{X}_f(g) = g(s)F(s) \text{ with } F(s) = i \int_0^s [\frac{d^2\Lambda}{dx^2}(x), \Lambda(x)] \, dx$$

is the Hamiltonian vector field that corresponds to f.

The integral curves $t \to g(s, t)$ of $\mathcal{X}_f$ are the solutions of the following partial differential equations

$$\frac{\partial g}{\partial t}(s, t) = g(t, s)i \int_0^s [\frac{d^2\Lambda}{dx^2}(x, t), \Lambda(x, t)] \, dx \qquad (37)$$

$$\frac{\partial g}{\partial s}(s, t) = g(t, s)\Lambda(t, s). \qquad (38)$$

Theorem 5 implies that the matrices $\Lambda(s, t)$ evolve according to

$$\frac{\partial \Lambda}{\partial t}(t, s) = i \left[\frac{\partial^2 \Lambda}{\partial s^2}, \Lambda(t, s) \right]. \qquad (39)$$

Equation (39 when expressed in terms of the coordinates $\lambda(s, t)$ of $\Lambda(s, t)$ relative to the basis of the Hermitian Pauli matrices becomes:

$$\frac{\partial \lambda}{\partial t}(t, s) = \lambda(t, s) \times \frac{\partial^2 \lambda}{\partial s^2}(s, t). \qquad (40)$$

Remark 2 *Equation (40) is well known in the literature in applied mathematics. L.D.Faddeev and L.A. Takhtajan refer to it as the continuous isotropic Heisenberg feromagnetic model([5], Part II.,Chapter 1) which they treat in an ad hoc manner as an equation in the space of Hermitian matrices. V.I. Arnold and B.Khesin ([2]) connect (40) to the filament equation which they furher consider as a special type of a Landau-Lifschitz equation on $so_3(R)$.*

Spherical Darboux curves. The derivation of the corresponding Hamiltonian equations on the sphere is similar to the preceding case except for the details related to the covariant derivative. Recall that the tangent space $T_X(Horiz(\mathcal{D}_s)(L)$ at an anchored horizontal Darboux curve $X(s)$ consists of tangent curves $v(s) = X(s)V(s)$ with $V(s)$ the solution of

$$\frac{dV}{ds}(s) = [\Lambda(s), V(s)] + U(s) \tag{41}$$

with $V(0) = 0$. The matrix $\Lambda(s)$ is the tangent vector of X, i.e.,

$$\frac{dX}{ds}(s) = X(s)\Lambda(s)$$

and $U(s)$ is a curve in $\mathfrak{k}$ subject to $U(0) = 0$ and $\langle \Lambda(s), U(s) \rangle = 0$.

Let $v(s) = X(s)V(s)$ be a fixed tangent vector at an anchored horizontal Darboux curve $X(s)$. To find the appropriate expression for $df_X(V)$ the directional derivative of f at X in the direction V, let $Y(s,t)$ denote a family of anchored horizontal Darboux curves such that $Y(s,0) = X(s)$ and such that $v(s) = \frac{\partial Y}{\partial t}(s,t)_{t=0}$.

Let $Z(s,t)$ denote the matrices defined by

$$\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t)$$

It follows that $\Lambda(s) = Z(s,0)$, and that $V(s)$ is the solution of (41) with $U(s) = \frac{\partial Z}{\partial t}(s,0)$.

Then,

$$\begin{aligned}
df_\Lambda(V) &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^L \langle \frac{\partial Z}{\partial s}(s,t), \frac{\partial Z}{\partial s}(s,t)\rangle\, ds|_{t=0} \\
&= \int_0^L \langle \frac{d\Lambda}{ds}(s), \frac{dU}{ds}(s)\rangle\, ds \\
&= -\int_0^L \langle \frac{d^2\Lambda}{ds^2}(s), U(s)\rangle\, ds + \langle \frac{d\Lambda}{ds}(s), U(s)\rangle_{s=0}^{s=L}.
\end{aligned}$$

Analogous to the hyperbolic case the boundary terms vanish in the frame-periodic case, and therefore

$$df_X(V) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, U(s) \right\rangle ds \tag{42}$$

The Hamiltonian vector field $\mathcal{X}_f$ that corresponds to f is of the form

$$\mathcal{X}_f(X)(s) = X(s)F(s)$$

for some curve $F(s) \in \mathfrak{k}$. Since $\mathcal{X}_f(X) \in T_X Horiz(\mathcal{PD})_s(L)$ $F(s)$ is the solution of

$$\frac{dF}{ds}(s) = [\Lambda(s), F(s)] + U_f(s)$$

with $F(0) = 0$ for some curve $U_f(s) \in \mathfrak{k}$ that satisfies

$$U_f(0) = 0 \text{ and } \langle \Lambda(s), U_f(s) \rangle = 0.$$

The curve $U_f(s)$ is determined by the symplectic form ω in Definition 2.11 through the usual relation

$$df_\Lambda(U) = -\int_0^L \langle \Lambda(s), [U_f(s), U(s)] \rangle \, ds. \tag{43}$$

The curve $U(s)$ satisfies $U(0) = 0$ and $\langle \Lambda(s), U(s) \rangle = 0$, and is otherwise arbitrary.

Equations (42) and (43) yield

$$\int_0^L \left\langle \frac{d^2\Lambda}{ds^2} - [\Lambda(s), U_f(s)], U(s) \right\rangle \, ds = 0. \tag{44}$$

The curve $U(s)$ can be written as $U(s) = [\Lambda(s), C(s)]$ where $C(s)$ is any curve that satisfies $C(0) = 0$. In that case equation (44) becomes

$$\int_0^L \left\langle [\frac{d^2\Lambda}{ds^2}, \Lambda] - [[\Lambda(s), U_f(s)], \Lambda], C(s) \right\rangle \, ds = 0. \tag{45}$$

Since $C(s)$ is arbitrary,

$$[\frac{d^2\Lambda}{ds^2}, \Lambda] - [[\Lambda(s), U_f(s)], \Lambda] = 0. \tag{46}$$

Lemma 3 implies that

$$[[\Lambda(s), U_f(s)], \Lambda] = U_f$$

and therefore,

$$U_f = -[\Lambda, \frac{d^2\Lambda}{ds^2}].$$

The integral curves $t \to X(s,t)$ of the vector field $\mathcal{X}_f$ are the solutions of

$$\frac{\partial X}{\partial t}(s,t) = X(s,t) F(s,t), \text{ and } \frac{\partial X}{\partial s} = X(s,t) \Lambda(s,t) \tag{47}$$

where $F(s,t)$ is the solution of

$$\frac{\partial F}{\partial s}(s,t) = [\Lambda(s,t), F(s,t)] - [\Lambda(s,t), \frac{d^2\Lambda}{ds^2}(s,t)].$$

But then according to Lemma 2

$$\frac{\partial \Lambda}{\partial t}(s,t) - \frac{\partial F}{\partial s}(s,t) + [\Lambda(s,t), F(s,t)] = 0$$

hence,

$$\frac{\partial \Lambda}{\partial t}(s,t) = -[\Lambda(s,t), \frac{\partial^2 \Lambda}{\partial s^2}(s,t)] = [\frac{\partial^2 \Lambda}{\partial s^2}(s,t), \Lambda(s,t)]. \tag{48}$$

Equation (48) describes the flow of the Hamiltonian vector field in the spherical case.

The reader should keep in mind however, that in the hyperbolic case Λ is Hermitian, while in the spherical case, Λ is skew-Hermitian: to pass from the hyperbolic case to the spherical case multiply $\Lambda(t,s)$ in equation (39) by i.

Definition 4.1 *Equations (39) and (48) shall be called Heisenberg's magnetic equations.*

4.2   The non-linear Schroedinger equation

Each solution $\Lambda(s,t)$ of Heisenberg's magnetic equation is generated by a family of periodic frames $t \to R(s,t)$ in $\mathfrak{k}$ through $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$ in the hyperbolic case, and through $\Lambda(s,t) = R(s,t)A_1R^*(s,t)$ in the spherical case. Curves $R(t,s)$ then evolve according to the differential equations:

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t), \text{ and } \frac{\partial R}{\partial t} = R(s,t)V(s,t)$$

for some curves of matrices $U(s,t)$ and $V(s,t)$ in $\mathfrak{k}$. Matrices $U(s,t)$ and $V(s,t)$ are not independent of each other since they conform to

$$\frac{\partial U}{\partial t}(s,t) - \frac{\partial V}{\partial t}(s,t) + [U(s,t), V(s,t)] = 0 \tag{49}$$

as demonstrated earlier in the paper (Lemma 2). Moreover, $V(0,t) = 0$ because the horizontal Darboux curves are anchored at $s = 0$. Equation (49) then implies that $\frac{dU}{dt}(0,t) = 0$.

Theorem 9 *Let $U(s,t) = \sum u_j(s,t)A_j$ generate a solution of Heisenberg's magnetic equation and let $u(s,t) = u_2(s,t) + iu_3(s,t)$. Then,*

$$\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(x,t)\,dx\right)$$

is a solution of the non-linear Schroedinger's equation

$$\frac{\partial}{\partial t}\psi(s,t) = i\frac{\partial^2\psi}{\partial s^2}(s,t) + i\frac{1}{2}|\psi(s,t)|^2\psi(s,t). \tag{50}$$

The following lemma, whose proof can be easily obtained from the Lie brackets in Table 1, is useful for the proof of the theorem.

Lemma 7

$$[A,[A,B]] = \langle A,B\rangle A - \langle A,A\rangle B$$

for any $A,B \in \mathfrak{h}$.

$$[[A,B],B] = \langle B,B\rangle A - \langle A,B\rangle B$$

for any $A \in su_2$ and $B \in \mathfrak{h}$.

Proof of the theorem The proof of the theorem will be done for the hyperbolic case although the arguments are the same in both cases, as will become clear below.

Since $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$, it follows that

$$\begin{aligned}
\frac{\partial \Lambda}{\partial t} &= \frac{\partial R}{\partial t}B_1R^* + RB_1\frac{\partial R^*}{\partial t} \\
&= R(VB_1 - B_1V)R^* \\
&= R[B_1, V]R^*
\end{aligned}$$

Similarly,

$$\frac{\partial \Lambda}{\partial s} = R[B_1, U]R^*, \quad \text{and} \quad \frac{\partial^2 \Lambda}{\partial s^2} = R\left([[B_1, U], U] + \left[B_1, \frac{\partial U}{\partial s}\right]\right) R^*$$

The fact that $\Lambda(s, t)$ evolves according to Heisenberg's magnetic equation implies that

$$[B_1, V] = i([[[B_1, U], U], B_1] + [[B_1, \frac{\partial U}{\partial s}], B_1]) \tag{51}$$

According to the relations in Lemma 7

$$[[B_1, U], U] = \langle U, B_1 \rangle U - \langle U, U \rangle B_1 = -(u_2^2 + u_3^2)B_1 + u_1 u_2 B_2 + B_3 u_1 u_3$$

, and therefore

$$[[[B_1, U], U], B_1] = u_1 u_3 A_2 - u_1 u_2 A_3.$$

Similarly,

$$[B_1, \frac{\partial U}{\partial s}] = \frac{\partial u_3}{\partial s} B_2 - \frac{\partial u_2}{\partial s} B_3, \text{ and } [[B_1, \frac{\partial U}{\partial s}], B_1] = -\frac{\partial u_3}{\partial s} A_3 - \frac{\partial u_2}{\partial s} A_2.$$

Then equation (51) reduces to

$$[B_1, V] = i(u_1(u_3 A_2 - u_2 A_3) - \frac{\partial u_3}{\partial s} A_3 - \frac{\partial u_2}{\partial s} A_2)$$

which can also be written as

$$[B_1, V] = -u_1(u_3 B_2 - u_2 B_3) + \frac{\partial u_3}{\partial s} B_3 + \frac{\partial u_2}{\partial s} B_2$$

because $A_j = iB_j, j = 1, 2, 3$.

Let $V(s, t) = v_1(s, t)A_1 + v_2(s, t)A_2 + v_3(s, t)A_3$. Then, $[B_1, V] = v_3 B_2 - v_2 B_3$, which together with the relations above yields

$$v_2 = -u_1 u_2 - \frac{\partial u_3}{\partial s}, \text{ and } v_3 = -u_1 u_3 + \frac{\partial u_2}{\partial s}. \tag{52}$$

Equation (52) written in terms of the complex functions $u = u_2 + iu_3$ and $v = v_2 + iv_3$ becomes

$$v(s, t) = -u_1(s, t)u(s, t) + i\frac{\partial u}{\partial s}(s, t). \tag{53}$$

The zero curvature equation $\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial V}{\partial s} + [U, V] = 0$ implies that

$$\frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s}(u_2^2 + u_3^2) \tag{54}$$

and that

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - \frac{\partial u_1}{\partial s}u - i(v_1 + u_1^2)u. \tag{55}$$

Equation (54) implies that

$$\frac{\partial}{\partial t} \int_0^s u_1(x,t)dx = v_1(s,t) + \frac{1}{2}(u_2^2(s,t) + u_3^2(s,t)) + c(t)$$

for some function $c(t)$. However this function must be equal to zero because the Darboux curves are anchored, and therefore $v_1(t,0) = 0$ and $u(t,0) = 0$.

Upon the substitution $v_1(s,t) = \frac{\partial}{\partial t} \int_0^s u_1(x,t)dx - \frac{1}{2}|u(s,t)|^2$ the equation (55) becomes

$$\frac{\partial u}{\partial t} + iu\frac{\partial}{\partial t} \int_0^s u_1(t,x)\, dx = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\left(-\frac{1}{2}|u|^2 + u_1^2\right)u. \qquad (56)$$

After the multiplication by $\exp\left(i\int_0^s u_1(x,t)\, dx\right)$ equation (56) can be expressed as

$$\frac{\partial}{\partial t}(u \exp\left(i\int_0^s u_1\, dx\right)) = (i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i(u_1^2 - \frac{1}{2}|u|^2))u\exp\left(i\int_0^s u_1\, dx\right)$$

because

$$\frac{\partial}{\partial t}(u(s,t)\exp\left(i\int_0^s u_1(x,t)\, dx\right)) = \exp\left(i\int_0^s u_1(x,t)\, dx\right)(\frac{\partial u}{\partial t} + iu\frac{\partial}{\partial t}\int_0^s u_1(x,t)\, dx).$$

The function
$$\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(t,x)\, dx\right)$$

satisfies
$$\frac{\partial \psi}{\partial s} = \left(\frac{\partial u}{\partial s} + iuu_1\right)\exp\left(i\int_0^s u_1(x,t)\, dx\right)$$

and
$$\frac{\partial^2 \psi}{\partial s^2} = \left(\frac{\partial^2 u}{\partial s^2} + 2iu_1\frac{\partial u}{\partial s} + iu\frac{\partial u_1}{\partial s} - u_1^2 u\right)\exp i\int_0^s u_1(x,t)\, dx.$$

It follows that
$$i\frac{\partial^2 \psi}{\partial s^2} = \left(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - iu_1^2 u\right)\exp i\int_0^s u_1(x,t\, dx$$

, and therefore
$$\frac{\partial}{\partial t}\psi(t,s) = i\frac{\partial^2 \psi}{\partial s^2} + i\frac{1}{2}|\psi|^2\psi.$$

∎

In the spherical case the evolution along Heisenberg's magnetic equation leads to

$$[A_1, V] = ([[[A_1, U], U], A_1] + [[A_1, \frac{\partial U}{\partial s}], A_1]).$$

The preceeding equation is the same as equation (51) because $A_1 = iB_1$. Therefore the calculations that led to the non-linear Schroedinger equation in the hyperbolic case are equally valid in the spherical case with the same end result.

The steps taken in the passage from Heisenberg's equation to the Schroedinger's equation are reversible. Any solution $\psi(s,t)$ of (50) generates a solution of the zero-curvature equation (as demonstrated in [5]). Simply let

$$U = \frac{1}{2} \begin{pmatrix} 0 & \psi \\ -\bar{\psi} & 0 \end{pmatrix} \text{ and } V = \frac{1}{2} \begin{pmatrix} i\frac{\partial}{\partial s}|\psi|^2 & i\psi(|\psi|^2 + c) \\ i\bar{\psi}|\psi|^2 & -i\frac{\partial}{\partial s}|\psi|^2 \end{pmatrix}.$$

Then curves $R(s,t)$ that evolve according to the differential equations:

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t), \ \frac{\partial R}{\partial t}(s,t) = R(s,t)V(s,t)$$

generate the solutions $\Lambda(s,t)$ of Heisenberg's magnetic equation through the familiar formulas $\Lambda(s,t) = R(s,t)B_1 R^*(s,t)$ or $\Lambda(s,t) = R(s,t)A_1 R^*(s,t)$ depending on the case.

To correlate the findings of this paper with the related existing literature, which almost exclusively deals with curves in $\mathbb{R}^3$, it seems appropriate to include a discussion of the only remaining simply connected three dimensional symmetric space, namely the Euclidean space.

4.3　Euclidean Darboux curves

The semidirect product of $\mathfrak{h}$ with K is the most convenient setting for comparisons with non-Euclidean Darboux curves. Recall that the semidirect product $V \rhd K$ of a vector space V and a group K which acts linearly on V consists of pairs (x, R) with $x \in V$ and $R \in K$. The group operation is given by $(x, R)(y, T) = (x + Ry, RT)$ for any elements (x, R) and (y, T). The Lie algebra $V \rhd \mathfrak{k}$ of the semidirect product $V \rhd K$ consists of pairs (a, A) with $a \in V$ and $A \in \mathfrak{k}$ with the Lie bracket $[(a, A), (b, B)] = (A(b) - B(a), [A, B])$. Both the vector space V and the Lie algebra $\mathfrak{k}$ can be embedded in the Lie algebra of the semidirect product via the embeddings $a \rightarrow (a, 0)$ and $A \rightarrow (0, A)$. With this identification, $V \rhd \mathfrak{k} = V \oplus \mathfrak{k}$ and

$$[V, V] = 0, \ [V, \mathfrak{k}] = V, \ [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}.$$

The group $K = SU_2$ acts linearly on the space of Hermitian matrices $\mathfrak{h}$ by $R(x) = RxR^*$ for $x \in V$ and $R \in K$, and $A(a) = [a, A]$ for $a \in \mathfrak{k}$ and $A \in \mathfrak{k}$. Therefore, the Lie bracket in $\mathfrak{h} \triangleright \mathfrak{k}$ is given by

$$[(a, A), (b, B)] = ([b, A] - [a, B], [A, B]).$$

The space of Hermitian matrices endowed with the metric induced by the trace form becomes a three dimensional Euclidean space, that will be denoted by E^3, while the semidirect product of K with E^3 will be denoted by $S_K(E_3)$.

The group $S_K(E_3)$ acts on E^3 by $(x, R)(y) = R(y) + x$ for each $(x, R) \in S_K(E_3)$ and each $y \in E^3$. The action is transitive, and K is equal to the isotropy group of the orbit through the origin $y = 0$. The Euclidean space E^3, when identified with the orbit through the origin becomes the coset space $S_K(E_3)/K$. The preceeding action extends to an action on the tangent bundle in which a tangent vector v at y is taken to the the tangent vector $R(v)$ at x under the action by an element $(x, R) \in S_K(E_3)$. The action on the tangent bundle extends further to an action on the orthonormal frame bundle of E^3 such that a frame (v_1, v_2, v_3) at a point $y \in E^3$ is taken to the frame $(R(v_1), R(v_2), R(v_3))$ at x under the action by an element $(x, R) \in S_K(E_3)$. The kernel of this action consists of $\pm I$, and hence $S_K(E_3)/\{\pm I\}$ can be identified with the positively oriented orthonormal frame bundle of E^3 as the orbit through the standard frame (B_1, B_2, B_3) at the origin.

In the left-invariant representation of the tangent bundle of $S_K(E_3)$, the tangent vectors at a point (x, R) are given by pairs $(R(a), RA)$ with $a \in E^3$ and $A \in \mathfrak{k}$. Hence curves $(x(s), R(s))$ in $S_K(E_3)$ are represented by differential equations

$$\tfrac{dx}{ds}(s) = R(s)(a(s)), \ \tfrac{dR}{ds}(s) = R(s)A(s). \tag{57}$$

The terminology concerning Darboux curves in non-Euclidean cases extends naturally to the Euclidean setting. In particular, curves $(x, R) \in S_K(E_3)$ are Euclidean Darboux curves if $\tfrac{dx}{ds}(s) = v_1(s) = R(s)(B_1)$, which holds whenever $a(s) = B_1$. Horizontal Euclidean Darboux curves are the projections $x(s)$ of Euclidean Darboux curves. Anchored horizontal Darboux curves are the solutions of

$$\frac{dx}{ds}(s) = R(s)(B_1), \ x(0) = 0$$

with $R(s)$ an arbitrary curve in K that originates at I when $s = 0$. Frame-periodic Darboux curves (x, R) conform to the periodicity of $R(s)$ with its period equal to the length of $x(s)$.

For any horizontal Darboux curve $x(s)$,

$$\frac{d^2x}{ds^2}(s) = R(s)([B_1, A(s)])$$

, and therefore

$$\kappa^2(s) = ||\frac{d^2x}{ds^2}(s)||^2 = u_2^2(s) + u_3^2(s)$$

where $A(s) = \sum u_i(s)A_i$. The frame $R(s)$ is a Serret-Frenet frame if $A(s) = \tau(s)A_1 + \kappa(s)A_3$, in which case the frame vectors $T(s), N(s), B(s)$ are given by

$$T(s) = R(s)(B_1), \ N(s) = R(s)(B_2), \ B(s) = R(s)(B_3).$$

The reader may easily verify that the tangent space at each anchored horizontal Darboux curve $x(s)$ consists of curves $v(s)$ such that

(a) $v(0) = \frac{dv}{ds}(0) = 0$, and

(b) $\langle \frac{dx}{ds}(s), \frac{dv}{ds}(s) \rangle = 0$.

The space of horizontal frame-periodic Euclidean Darboux curves inherits the symplectic structure given by Definition 3.11. This symplectic structure is isomorphic to the structure used by J. Millson and B.A. Zombro in ([17]) as can be easily seen from Lemma 6. More precisely, in the Millson-Zombro paper the Euclidean space E^3 is identified with $so_3(R)$ which is isomorphic to su_2, and their symplectic form is identical to the one given by equation (22).

It can be shown by arguments identical to the ones already presented in this paper that the Hamiltonian flow induced by the function $f(x(s)) = \frac{1}{2} \int_0^l \kappa^2(s) \, ds$ leads to Heisenberg's magnetic equation, and that the passage to the non-linear Schroedinger's equation is the same as the one presented for the non-Euclidean cases.

The present formalism clarifies Hasimoto's first observation that $\psi = \kappa \exp\left(i \int \tau \, dx\right)$ of a curve $\gamma(s, t)$ that satisfies the filament equation

$$\frac{\partial \gamma}{\partial t}(s, t) = \kappa(s, t)B(s, t) \tag{58}$$

is a solution of the non-linear Schroedinger equation. In this notation, it is understood that $t \rightarrow \gamma(s, t)$ denotes a family of curves in R^3 parametrized by t and that $B(s, t)$ denotes the binormal vector along the curve $s \rightarrow \gamma(s, t)$.

When the solution curves of the filament equation are restricted to curves parametrized by arc-length,i.e., to curves $\gamma(s)$ such that $||\frac{d\gamma}{ds}(s)|| = 1$ then

$$T(t, s) = \frac{\partial \gamma}{\partial s}(t, s) \text{ and } \frac{dT}{ds}(s) = \kappa(s, t)N(s, t) = \frac{\partial^2 \gamma}{\partial s^2}(s, t).$$

Moreover, $B(s, t) = T(s, t) \times N(s, t)$. It then follows that in the space of arc-length parametrized curves the filament equation can be written as

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2}. \tag{59}$$

For each solution curve $\gamma(s, t)$ of (59) the tangent vector $T(s, t)$ satisfies

$$\frac{\partial T}{\partial t} = T \times \frac{\partial^2 T}{\partial s^2} \tag{60}$$

as can be easily verified by differentiating with respect to s.

Any solution $T(s, t)$ of the preceeding equation may be interpreted as the coordinate vector of $\Lambda(s, t)$ relative to an orthonormal basis in the Cartan space $\mathfrak{p}$. Then, $\Lambda(s, t)$ evolves according to

$$\frac{\partial \Lambda}{\partial t}(s, t) = i[\frac{\partial^2 \Lambda}{\partial s^2}(s, t), \Lambda(s, t)]$$

if $\mathfrak{p} = \mathfrak{h}$, and evolves according to

$$\frac{\partial \Lambda}{\partial t}(s, t) = [\frac{\partial^2 \Lambda}{\partial s^2}(s, t), \Lambda(s, t)]$$

if $\mathfrak{p} = \mathfrak{k}$.

The function $\psi(s, t) = u(s, t) \exp\left(i \int_0^s u_1(x, t)\, dx\right)$ associated with the frame $R(s, t)$ that defines $\Lambda(s, t)$ a solution of the non-linear Schroedinger's equation independently of the choice of the symmetric space (Theorem 9). When the frame $R(s, t)$ is a Serret-Frenet frame then:

$u_1 = \tau$, $u_2 = -\kappa$, $u_3 = 0$ in the hyperbolic case,
$u_1 = \tau + \frac{1}{2}$, $u_2 = 0$, $u_3 = \kappa$ in the spherical case, and
$u_1 = \tau$, $u_2 = 0$, $u_3 = \kappa$ in the Euclidean case.
In all cases,

$$\psi(s, t) = u(s, t) \exp\left(i \int_0^s u_1(x, t)\, dx\right) = (\exp i\theta)(\kappa(s, t) \exp i \int_0^s \tau(x, t)\, dx))$$

for some angle θ, and therefore, Hasimoto's function $\kappa(s, t) \exp i \int_0^s \tau(x, t)\, dx)$ is a solution of the non-linear Schroedinger's equation since the latter is invariant under circular rotations. The geometry of the underlying space becomes visible only when the integration of the Hamiltonian equations is carried out on the full tangent bundle of the Lie group and not just on the part of the equations that resides in the Lie algebra $\mathfrak{g}$.

5   Elastic curves and solitons

For mechanical systems the Hamiltonian function represents the total energy of the system and its critical points correspond to the equilibrium configurations. In an infinite-dimensional setting the behaviour of a Hamiltonian system at a critical point of a Hamiltonian seems not to lend itself to such simple characterizations.

For the Hamiltonian function $f = \frac{1}{2} \int_0^L k^2\, ds$ the critical points are the elastic curves. The solutions of the associated Hamiltonian system that originate at an elastic curve, instead of being stationary, form travelling waves known as solitons. Soliton solutions of either Heisenberg's magnetic equation or the non-linear Schroedinger's equation are waves that travel at constant speeds with an elastic curve at their wave fronts. To explain these statements in some detail it will be necessary to make a small detour into the geometry of elastic curves.

5.1 Elastic curves and their Hamiltonian systems

To maintain continuity with the material already presented and yet to keep the detour at a minimum, the discussion will be confined to the semidirect product $S_K(E^3)$ and $SL_2(C)$. The spherical case $K \times K$ requires adjustments in notation but is otherwise similar to the other two cases (as demonstrated in [11],[12],[13]).

For notational simplicity G_ϵ will denote $S_K(E^3)$ for $\epsilon = 0$, and $SL_2(C)$ for $\epsilon = -1$. The Lie algebra of G_ϵ will be denoted by $\mathfrak{g}_\epsilon$. As sets $\mathfrak{g}_0 = \mathfrak{g}_{-1}$, but as algebras they are different. Their Lie brackets conform to the following table

$[\ ,\ ]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	$-\epsilon A_3$	ϵA_2
B_2	B_3	0	$-B_1$	ϵA_3	0	$-\epsilon A_1$
B_3	$-B_2$	B_1	0	$-\epsilon A_2$	ϵA_1	0

Table 2

Definition 5.1 *The problem of finding the minimum of the integral*

$$\frac{1}{2} \int_0^L (u_2^2(s) + u_3^2(s))\, ds$$

over all curves $g(s)$ in G_ϵ that are the solutions of

$$\frac{dg}{ds}(s) = g(s)(B_1 + u_2(s)A_2 + u_3(s)A_3) \tag{61}$$

and satisfy fixed boundary conditions at $s = 0$ and $s = L$ shall be called the elastic problem on G_ϵ.

Definition 5.2 *The projections $x(s) = \pi_\epsilon(g(s))$ of the "extremal curves" $g(s)$ on the underlying space G_ϵ/K are called elastic curves.*

It is known that the elastic problem has a solution for any pair of boundary points provided that L is sufficiently large. Since the elastic problem is left-invariant, the initial point can always be taken at the identity. It is evident from the first part of the paper that

$$\frac{1}{2} \int_0^L (u_2^2(s) + u_3^2(s))\, ds = \frac{1}{2} \int_0^L \kappa^2(s)\, ds$$

where $\kappa(s)$ is the curvature of the projected curve $\pi_\epsilon(g(s))$. The set of curves (61) may be considered as a "reduced" Darboux space for the function $f(g) = \frac{1}{2} \int_0^L \kappa^2(s)\, ds$ for the following reasons:

For any Darboux curve $g(s)$ that is a solution of

$$\frac{dg}{ds}(s) = g(s)(B_1 + u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3)$$

$g_0(s) = g(s) \exp\left(-A_1 \int_0^s u_1(x)\,dx\right)$ projects onto the same base point as $g(s)$ and satisfies (61). Consequently,

$$f(g(s)) = f(g_0(s)).$$

It should be noted that when the solution $R(s)$ of the equation

$$\frac{dR}{ds}(s) = R(s)(u_2(s)A_2 + u_3(s)A_3)$$

is periodic, then $h(s) = g(s)R^*(s)$ is an anchored frame-periodic curve, and therefore, $u(s) = u_2(s) + iu_3(s)$ is a solution of the non-linear Schroedinger's equation (Theorem 9).

The solutions of the elastic problem seen from control theoretic perspectives in accordance with the Maximum Principle are confined to the projections of integral curves, called extremal curves, of a Hamiltonian system on the cotangent bundle T^*G_ϵ of G_ϵ ([11]).

To take advantage of the left-invariant symmetries, the cotangent bundle T^*G_ϵ will be represented via the left translations as $G_\epsilon \times \mathfrak{g}_\epsilon^*$, where $\mathfrak{g}_\epsilon^*$ denotes the dual of $\mathfrak{g}_\epsilon$. Linear functions in $\mathfrak{g}_\epsilon^*$ will be represented by the coordinate functions $h_1, h_2, h_3, H_1, H_2, H_3$ relative to the dual basis $B_1^*, B_2^*, B_3^*, A_1^*, A_2^*, A_3^*$ defined by the Pauli matrices .

An easy application of the Maximum Principle shows that the regular extremal curves of the elastic problem are the projections of the integral curves of the Hamiltonian vector field $\vec{H}$ defined by the Hamiltonian function

$$H = \tfrac{1}{2}(H_2^2 + H_3^3) + h_1. \tag{62}$$

The extremal functions u_2 and u_3 are of the form

$$u_2 = H_2,\ u_3 = H_3. \tag{63}$$

Remark 3 *The abnormal extremal curves shall be ignored. It is known ([10]) that the abnormal extremal curves exist, and may or may not project onto an optimal solutions of the elastic problem. However, the optimal solutions that are the projections of abnormal extremal curves are also the projections of regular extremal curves, and therefore could be ignored, at least as far as optimality is concerned.*

The most direct way to get the equations of $\vec{H}$ is via the Poisson brackets involving the variables $h_1, h_2, h_3, H_1, H_2, H_3$. The Poisson brackets of these variables are isomorphic to the Lie brackets in Table 2, and are reproduced for the convenience of the reader in Table 3 below.

$\{\ ,\ \}$	H_1	H_2	H_3	h_1	h_2	h_3
H_1	0	$-H_3$	H_2	0	$-h_3$	h_2
H_2	H_3	0	$-H_1$	h_3	0	$-h_1$
H_3	$-H_2$	H_1	0	$-h_2$	h_1	0
h_1	0	$-h_3$	h_2	0	$-\epsilon H_3$	ϵH_2
h_2	h_3	0	$-h_1$	ϵH_3	0	$-\epsilon H_1$
h_3	$-h_2$	h_1	0	$-\epsilon H_2$	ϵH_1	0

Table 3

Therefore, the Hamiltonian equations are given by:

$$
\begin{aligned}
\frac{dH_1}{ds} &= \{H_1, H\} = H_2\{H_1, H_2\} + H_3\{H_1, H_3\} + \{H_1, h_1\} = 0 \\
\frac{dH_2}{ds} &= \{H_2, H\} = H_3\{H_2, H_3\} + \{H_2, h_1\} = -H_3 H_1 + h_3 \\
\frac{dH_3}{ds} &= \{H_3, H\} = H_2\{H_3, H_2\} + H_3\{H_3, H_3\} + \{H_3, h_1\} = H_2 H_1 - h_2 \\
\frac{dh_1}{ds} &= \{h_1, H\} = H_2\{h_1, H_2\} + H_3\{h_1, H_3\} + \{h_1, h_1\} = H_3 h_2 - H_2 h_3 \\
\frac{dh_2}{ds} &= \{h_2, H\} = H_2\{h_2, H_2\} + H_3\{h_2, H_3\} + \{h_2, h_1\} = -H_3 h_1 + \epsilon H_3 \\
\frac{dh_3}{ds} &= H_2\{h_3, H_2\} + H_3\{h_3, H_3\} + \{h_3, h_1\} = \{h_3, H\} = H_2 h_1 - \epsilon H_2.
\end{aligned}
\tag{64}
$$

It follows by an easy calculation that

$$
I_1 = h_1^2 + h_2^2 + h_3^2 + \epsilon(H_1^2 + H_2^2 + H_3^2)
$$

and

$$
I_2 = h_1 H_1 + h_2 H_2 + h_3 H_3
$$

are the constants of motion for (64). Together with H_1 and H, I_1 and I_2 account for four independent constants of motion, and therefore, (64) is completely integrable.

Any solution $h_1(s), h_2(s), h_3(s), H_1(s), H_2(s), H_3(s)$ of (64) defines complex functions $u(s) = H_2(s) + iH_3(s)$ and $w(s) = h_2(s) + ih_3(s)$. Then, $u(s)$ generates a soliton solution if there exists a real number ξ such that $\psi(s, t) = u(s + \xi t)$ is a solution of the non-linear Schroedinger's equation.

It follows from equations (64) that

$$
\frac{du}{ds}(s) = iH_1 u(s) - iw(s), \text{ and } \frac{dw}{ds} = i(h_1 - \epsilon)u(s). \tag{65}
$$

Therefore,

$$
\frac{\partial \psi}{\partial t} = i\xi(H_1\psi - w), \text{ and } \frac{\partial^2 \psi}{\partial s^2} = -H_1^2\psi + H_1 w + (h_1 - \epsilon)\psi.
$$

Since $H = \frac{1}{2}|\psi|^2 + h_1$,

$$
\begin{aligned}
-i\frac{\partial \psi}{\partial t} - (\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi) &= \xi(H_1\psi - w) - (-H_1^2\psi + H_1 w + (h_1 - \epsilon)\psi + \psi(H - h_1)) \\
&= -(\xi + H_1)w + (\xi H_1 + H_1^2 + \epsilon - H)\psi.
\end{aligned}
$$

The preceding quantity is a solution of the non-linear Schroedinger's equation whenever

$$
\xi = -H_1 \text{ and } H = \epsilon. \tag{66}
$$

Remark 4 *The reader may verify that (66) is a necessary condition for the existence of solitons for all three cases. For the spherical case $\epsilon = 1$ and H_1 and H are the same as in the cases discussed above.*

To show that periodic solutions $u(s)$ exist on the energy level $H = \epsilon$ requires explicit formula for $u(s)$ in terms of the remaining constants of motion I_1 and I_2.

To begin with, note that

$$(H_2 h_3 - H_3 h_2)^2 + (H_2 h_2 + H_3 h_3)^2 = (H_2^2 + H_3^2)(h_2^2 + h_3^2).$$

Then,

$$
\begin{aligned}
(\tfrac{d}{ds} h_1)^2 &= (H_2 h_3 - H_3 h_2)^2 = (H_2^2 + H_3^2)(h_2^2 + h_3^2) - (H_2 h_2 + H_3 h_3)^2 \\
&= (H_2^2 + H_3^2)(I_1 - \epsilon(H_1^2 + H_2^2 + H_3^2) - h_1^2) - (I_2 - h_1 H_1)^2 \\
&= 2(H - h_1)(I_1 - \epsilon H_1^2 - 2\epsilon(H - h_1) - h_1^2) - (I_2 - h_1 H_1)^2 \\
&= 2h_1^3 + c_1 h_1^2 + c_2 h_1 + c_3
\end{aligned}
\tag{67}
$$

where c_1, c_2, c_3 are the constants of motion given by the following expressions

$$c_1 = -(H_1^2 - 2H - 4\epsilon), c_2 = (2I_2 H_1 - 2\epsilon H_1^2 + 4\epsilon H - 2I_1), c_3 = 2H(I_1 - \epsilon H_1^2 - 2\epsilon H) - I_2^2.$$

Therefore, $h_1(s)$ is expressed in terms of elliptic functions, and since $k^2 = H_2^2 + H_3^2 = 2(H - h_1)$ the same can be said for the curvature of the projected elastic curve. The remaining variables $u = H_2 + iH_3$ and $w = h_2 + ih_3$ can be integrated in terms of two angles θ and ϕ defined as follows:

$$
\begin{aligned}
I_1 &= h_1^2 + |w|^2 + \epsilon(H_1^2 + |u|^2) = h_1^2 + |w|^2 + \epsilon(H_1^2 + 2(H - h_1)) \\
&= (h_1 - \epsilon)^2 + |w|^2 + \epsilon H_1^2 + 2\epsilon H - \epsilon^2
\end{aligned}
$$

and therefore,

$$(h_1 - \epsilon)^2 + |w|^2 = J^2 \tag{68}$$

where J^2 denotes $I_1 - \epsilon H_1^2 - 2\epsilon H + \epsilon^2$. Since J is constant along each extremal trajectory, equation (68) defines a sphere along each extremal curve. The angles θ and ϕ are defined on that sphere by

$$(h_1(s) - \epsilon) = J \cos \theta(s) \text{ and } w(s) = J \sin \theta(s) e^{i\phi(s)}. \tag{69}$$

It follows that

$$\tfrac{dh_1}{ds} = -J \sin \theta \tfrac{d\theta}{ds}, \text{ and } \tfrac{dw}{ds} = w(\tfrac{\cos\theta}{\sin\theta}\tfrac{d\theta}{ds} + i\tfrac{d\phi}{ds}). \tag{70}$$

Furthermore,

$$
\begin{aligned}
\frac{u}{w} = \frac{u\bar{w}}{|w|^2} &= \frac{H_2 h_2 + H_3 h_3 + i(H_3 h_2 - H_2 h_3)}{J^2 - (h_1 + \epsilon)^2} \\
&= \frac{I_2 - h_1 H_1 + i\frac{dh_1}{ds}}{J^2 \sin^2 \theta} \\
&= \frac{I_2 - h_1 H_1}{J^2 \sin^2 \theta} - \frac{i}{J \sin \theta}\frac{d\theta}{ds}.
\end{aligned}
$$

Therefore,

$$w(\frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds} + i\frac{d\phi}{ds}) \;=\; \frac{dw}{ds}$$

$$= \; i(h_1 - \epsilon)u = (iJ\cos\theta\frac{(I_2 - h_1 H_1)}{J^2\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds})w$$

hence,

$$\frac{d\phi}{ds} = J\cos\theta\frac{(I_2 - h_1 H_1)}{J^2\sin^2\theta} = \frac{J\cos\theta(I_2 - \epsilon H_1 - H_1 J\cos\theta)}{J^2\sin^2\theta}. \qquad (71)$$

Equation (67) can be also written as

$$(\frac{dh_1}{ds})^2 = 2(H - h_1)|w|^2 - (I_2 - H_1 h_1)^2.$$

The substitutions from (69) and (70) in the preceeding equation define θ as the solution of the following differential equation

$$(\frac{d\theta}{ds})^2 = 2(H - \epsilon - J\cos\theta) - \frac{(I_2 - H_1(\epsilon + J\cos\theta))^2}{J^2\sin^2\theta}. \qquad (72)$$

It follows from (69) and the equation that follows that both $u(s)$ and $w(s)$ are determined by the angles $\phi(s)$ and $\theta(s)$. We now return to the question of periodicity of $u(s)$. Evidently, both u and w are periodic whenever $\phi(0) = \phi(L)$ and $\theta(0) = \theta(L)$. Soliton solutions propagate with speed $\psi = H_1$ on the energy level $H = \epsilon$. On this energy level $\phi(0) = \phi(L)$ and $\theta(0) = \theta(L)$ if and only if

$$\int_0^L \frac{J\cos\theta(I_2 + H_1 - H_1 J\cos\theta)}{J^2\sin^2\theta}\, ds = 0 \qquad (73)$$

where θ denotes a closed solution of the equation

$$(\frac{d\theta}{ds})^2 = -2J\cos\theta - \frac{(I_2 - H_1(\epsilon + J\cos\theta))^2}{\sin^2\theta}. \qquad (74)$$

It is known that there are are infinitely many closed solutions for suitable constants I_1, I_2, H_1 (for instance, ([9])) however, such calculations will not be presented here.

6 Complete Integrability

There are further connections between elasic curves and solutions of the non-linear Schroedinger equation that were first noticed by J. Langer and R. Perline, namely that some of the integrals of motion for the elastic curves correspond to the integrals of motion for the non-linear Schroedinger's equation ([14]). We will illustrate this phenomenon by showing that the function $f(\gamma) = \int_0^L k^2\tau\, ds$ is an integral of motion

for Heisenberg's magnetic equation, while the quantity $k^2(s)\tau(s)$ is a constant of motion for the elastic problem.

We shall first show that $k^2\tau$ is a constant of motion for the elastic curves. The Serret-Frenet frame and the frame in Definition 5.1 rotate around each other in the plane perpendicular to the tangent vector. If we denote by β the angle through which the frame in Definition 5.1 rotates relative to the Serret-Frenet frame then, according to the formula in ([12], p461)

$$\frac{d\beta}{ds}(s) = \tau \text{ and } \tan\beta = -\frac{H_2}{H_3}.$$

It follows that

$$
\begin{aligned}
\sec^2\beta\frac{d\beta}{ds} &= \frac{(H_2\frac{dH_3}{ds} - H_3\frac{dH_2}{ds})}{H_3^2} \\
&= \frac{H_2(H_2H_1 - h_2) - H_3(-H_3H_1 + h_3)}{H_3^2} \\
&= \frac{H_1(H_2^2 + H_3^2) - H_2h_2 - H_3h_3}{H_3^2} \\
&= \frac{H_1(H_2^2 + H_3^2) - (I_2 - H_1h_1)}{H_3^2} \\
&= \frac{2H_1H - I_2}{H_3^2}.
\end{aligned}
$$

Since

$$\sec^2\beta = (H_2^2 + H_3^2)/H_3^2 = \kappa^2/H_3^2$$

it follows that

$$\kappa^2\tau = 2H_1H - I_2 = constant.$$

Theorem 10 *(a) The Hamiltonian flow of $f_1 = \int_0^l \kappa^2(s)\tau(s)\,ds$ is given by*

$$\frac{\partial\Lambda}{\partial t} = 2(\dddot\Lambda - \langle\dddot\Lambda, \Lambda\rangle\Lambda) - 3\left\langle\Lambda, \ddot\Lambda\right\rangle\dot\Lambda). \tag{75}$$

(b) $f_0 = \frac{1}{2}\int_0^L \kappa^2(s)\,ds$ and f_1 Poisson commute.
Proof.
The first part of the proof consists in showing that

$$\kappa^2\tau = -i\left\langle[\Lambda, \frac{d\Lambda}{ds}], \frac{d^2\Lambda}{ds^2}\right\rangle.$$

Suppose now that $T(s) = \Lambda(s)$ denotes the Hermitian matrix that corresponds to the tangent vector of a horizontal Darboux curve that projects onto a curve $\gamma \in \mathbb{H}^3$.

Then $N(s)$ and $B(s)$, the matrices that correspond to the normal and the binormal vectors, are given by

$$N = \frac{1}{\kappa}\frac{d\Lambda}{ds} \text{ and } B(s) = \frac{1}{i}[T(s), N(s)] = -\frac{i}{\kappa}[\Lambda, \frac{d\Lambda}{ds}].$$

According to the Serret-Frenet equations $\frac{dN}{ds} = -k\Lambda + \tau B$. Therefore,

$$\tau = \left\langle \frac{dN}{ds}, B \right\rangle = -i\left\langle -\frac{1}{k^2}\frac{dk}{ds}\frac{d\Lambda}{ds} + \frac{1}{k}\frac{d^2\Lambda}{ds^2}, \frac{1}{k}[\Lambda, \frac{d\Lambda}{ds}] \right\rangle.$$

It follows that

$$k^2\tau = -i\left\langle [\Lambda, \frac{d\Lambda}{ds}], \frac{d^2\Lambda}{ds^2} \right\rangle.$$

Let $V(s)$ be an arbitrary tangent vector at a frame-periodic horizontal Darboux curve $g(s)$. Then the directional derivative of f_1 at g in the direction V is given by the following expression:

$$df_1(V) = -i\frac{\partial}{\partial t}\int_0^L \langle [Z(s,t), \frac{\partial Z}{\partial s}(s,t)], \frac{\partial^2 Z}{\partial s^2}(s,t) \rangle \, ds|_{t=0}$$

where the dots indicate derivatives with respect to s, and where $Z(s,t)$ denotes a field of Hermitian matrices such that

$$Z(s, 0 = \Lambda(s) \text{ and } \frac{\partial Z}{\partial t}(s,0) = V(s).$$

It follows that

$$\begin{aligned}
df_1(V) &= -i\int_0^L \left\langle \dot{V}, [\Lambda, \dot\Lambda] \right\rangle + \left\langle \ddot\Lambda, [\dot{V}, \dot\Lambda] \right\rangle + \left\langle \ddot\Lambda, [\Lambda, \ddot{V}] \right\rangle \, ds \\
&= -i\int_0^L 2\left\langle [\ddot\Lambda, \Lambda], \ddot{V} \right\rangle - \left\langle [\ddot\Lambda, \dot\Lambda], \dot{V} \right\rangle \, ds \\
&= i\int_0^L \left\langle 2(\frac{d}{ds}([\ddot\Lambda, \Lambda]) + [\ddot\Lambda, \dot\Lambda]), \dot{V} \right\rangle \, ds \\
&= i\int_0^L \left\langle 2[\dddot\Lambda, \Lambda]) + 3[\ddot\Lambda, \dot\Lambda], \dot{V} \right\rangle \, ds.
\end{aligned}$$

Let $V_1(s)$ denote the Hermitian matrix such that

$$df_1(V) = \omega_\Lambda(V_1, V).$$

Then,

$$i\int_0^L \left\langle 2[\dddot\Lambda, \Lambda] + 3[\ddot\Lambda, \dot\Lambda], \dot{V} \right\rangle \, ds = \frac{1}{i}\int_0^L \langle [\Lambda, \dot{V}_1], \dot{V} \rangle \, ds$$

which implies

$$\int_0^L \langle [\Lambda, \dot{V}_1] + 2[\dddot{\Lambda}, \Lambda] + 3[\ddot{\Lambda}, \dot{\Lambda}], \dot{V} \rangle \, ds = 0.$$

When $\dot{V} = [\Lambda, C]$ the above becomes

$$\int_0^L \langle [[\Lambda, \dot{V}_1], \Lambda] + 2[[\dddot{\Lambda}, \Lambda], \Lambda] + 3[[\ddot{\Lambda}, \dot{\Lambda}], \Lambda], C(s) \rangle \, ds = 0.$$

Since $C(s)$ can be an arbitrary curve with $C(0) = 0$ the preceeding integral equality reduces to

$$[[\Lambda, \dot{V}_1], \Lambda] + 2[[\dddot{\Lambda}, \Lambda], \Lambda]) + 3[[\ddot{\Lambda}, \dot{\Lambda}], \Lambda] = 0.$$

The Lie bracket relations in Lemma 3 imply that

$$\dot{V}_1 + 2(\langle \dddot{\Lambda}, \Lambda \rangle \Lambda - \dddot{\Lambda}) + 3\langle \ddot{\Lambda}, \Lambda \rangle \dot{\Lambda} = 0.$$

Now it follows by the arguments used earlier in the paper that the Hamiltonian flow X_{f_1} satisfies

$$\frac{\partial \Lambda}{\partial t} = 2(\dddot{\Lambda} - \langle \dddot{\Lambda}, \Lambda \rangle \Lambda) - 3\left\langle \Lambda, \ddot{\Lambda} \right\rangle \dot{\Lambda}).$$

Thus part (a) is proved.

To prove part (b) it is required to show that the Poisson bracket of f_0 and f_1, given by the formula

$$\{f_0, f_1\}(\Lambda) = \omega_\Lambda(V_0(\Lambda), V_1(\Lambda)) = \frac{1}{i} \int_0^L \langle \Lambda(s), [\dot{V_0}(s), \dot{V}_1(s)] \, ds$$

with $\dot{V}_0(\Lambda) = i[\ddot{\Lambda}, \Lambda]$ and $\dot{V}_1 = -(2(\langle \dddot{\Lambda}, \Lambda \rangle \Lambda - \dddot{\Lambda}) + 3\langle \ddot{\Lambda}, \Lambda \rangle \dot{\Lambda})$, is equal to 0.

An easy calculation shows that

$$[\dot{V_0}, \dot{V}_1] = i(2(\left\langle \dddot{\Lambda}, \ddot{\Lambda} \right\rangle) - \left\langle \Lambda, \ddot{\Lambda} \right\rangle \langle \dddot{\Lambda}, \Lambda \rangle) - 3\left\langle \ddot{\Lambda}, \Lambda \right\rangle \left\langle \ddot{\Lambda}, \dot{\Lambda} \right\rangle)\Lambda.$$

Hence,

$$\{f_0, f_1\} = \int_0^L (2\left\langle \dddot{\Lambda}, \ddot{\Lambda} \right\rangle - 2\langle \Lambda, \dddot{\Lambda} \rangle \left\langle \Lambda, \ddot{\Lambda} \right\rangle - 3\left\langle \Lambda, \ddot{\Lambda} \right\rangle \left\langle \dot{\Lambda}, \ddot{\Lambda} \right\rangle) \, ds.$$

The integral of the first term is zero because $2\left\langle \dddot{\Lambda}, \ddot{\Lambda} \right\rangle = \frac{d}{ds}\left\langle \ddot{\Lambda}, \ddot{\Lambda} \right\rangle$.

Since $2\left\langle \Lambda, \dddot{\Lambda} \right\rangle \langle \Lambda, \ddot{\Lambda} \rangle = \frac{d}{ds}\left\langle \Lambda, \ddot{\Lambda} \right\rangle^2 - 2\left\langle \Lambda, \ddot{\Lambda} \right\rangle \left\langle \dot{\Lambda}, \ddot{\Lambda} \right\rangle$, the remaining integrand reduces to one term $-\left\langle \Lambda, \ddot{\Lambda} \right\rangle \left\langle \dot{\Lambda}, \ddot{\Lambda} \right\rangle$. But then $\frac{1}{4}\frac{d}{ds}\left\langle \dot{\Lambda}, \dot{\Lambda} \right\rangle^2 = \left\langle \Lambda, \ddot{\Lambda} \right\rangle \left\langle \dot{\Lambda}, \ddot{\Lambda} \right\rangle$ because $\left\langle \dot{\Lambda}, \dot{\Lambda} \right\rangle = -\left\langle \Lambda, \ddot{\Lambda} \right\rangle$, and part (b) is proved. ∎

Theorem 11 *Suppose that $\Lambda(s,t) = R(s,t)B_1 R^*(s,t)$ evolves according to the equation (75) where $R(s,t)$ is the solution of*

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)\begin{pmatrix} 0 & u(s,t) \\ -\bar{u}(s,t) & 0 \end{pmatrix}, \; R(0,t) = I.$$

Then, $u(s,t)$ is a solution of

$$\frac{\partial u}{\partial t} - 3|u|^2 \frac{\partial u}{\partial s} - 2\frac{\partial^3 u}{\partial s^3} = 0. \tag{76}$$

This theorem is proved by a calculation similar to the one used in the proof of Theorem 9, the details of which will not be reproduced here.

Equation (76) is similar to the modified Korteweg-de Vries equation (R.Abraham and J. Marsden ([1]))

$$v_t - 6v^2 v_x + v_{xxx} = 0 \tag{77}$$

with some notable differences. Equation (76) is a complex equation while the modified Korteweg-de Vries equation is a real equation. Because of the difference in sign in front of the third derivative it is not apparent that equation (76) is the complexification (modulo some homothetical transformation) of the equation (77). It remains an intriguing question if there are any connections between the Korteweg-de Vries equation and the elastic curves.

Functions f_0 and f_1 also appear in a paper on integrability of the non-linear Schroedinger's equation by C. Shabat and V. Zacharov ([18]), as first noticed by J. Langer and R. Perline ([14]), but in a completely different context. The first two integrals of motion in the paper of Shabat and Zacharov are up to the constant factors given by the following integrals:

$$C_1 = \int_{-\infty}^{\infty} |u(s,t)|^2 \, ds, \; C_2 = \int_{-\infty}^{\infty} (u(s,t))\dot{\bar{u}}(s,t)) - \bar{u}(s,t)\dot{u}(s,t)) \, ds$$

where they are interpreted as the number of particles and the momentum. To see that C_1 and C_2 are in exact correspondence with functions f_0 and f_1 assume that the Darboux curves are expressed by reduced frames $R(s)$ as in Definition 5.1 i.e., as the solutions of

$$\frac{dR}{ds}(s) = R(s)U(s) \text{ with } U(s) = u_2(s)A_2 + u_3(s)A_3$$

Then,

$$f_0 = \frac{1}{2}\int_0^L \|\dot{\Lambda}(s)\|^2 \, ds = \frac{1}{2}\int_0^L \|[B_1, U(s)]\|^2, ds = \frac{1}{2}\int_0^L |u(s)|^2 \, ds.$$

Hence, C_1 corresponds to $\int_0^L \kappa^2(s) \, ds$. Furhermore, $f_1 = \int_0^L k^2 \tau \, ds$ can be written as

$$f_1 = \frac{1}{2i}\int_0^L (u(s)\dot{\bar{u}}(s) - \bar{u}(s)\dot{u}(s)) \, ds$$

because

$$i\left\langle \Lambda, [\dot{\Lambda}, \ddot{\Lambda}] \right\rangle = \left\langle [[B_1, U], [B_1, \dot{U}]], B_1 \right\rangle = Im\bar{u}\dot{u},$$

and therefore f_1 corresponds to C_2.

In the language of mathematical physics the vector $\int_0^L \Lambda(s)\,ds$ is called the total spin ([5]). In this paper it appears as the moment map discussed in the previous section. It is a conserved quantity since the Hamiltonian is invariant under the action of SU_2. This fact can be verified directly as follows:

$$\frac{\partial}{\partial t}\int_0^L \Lambda(t,s)\,ds = \int_0^L \frac{\partial \Lambda}{\partial t}(t,s)\,ds = i\int_0^L [\frac{\partial^2 \Lambda}{\partial s^2}, \Lambda]\,ds = i\int_0^L \frac{\partial}{\partial s}[\Lambda, \dot{\Lambda}]\,ds = 0.$$

The third integral of motion C_3 in ([18]), called the energy, is given by

$$C_3 = \int_{-\infty}^{\infty}(|\frac{\partial u}{\partial s}(s,t)|^2 - \frac{1}{4}|u(s,t)|^4)\,ds.$$

It corresponds to the function

$$f_2 = \int_0^L (||\ddot{\Lambda}(s)||^2 - \frac{5}{4}||\dot{\Lambda}(s)||^4)\,ds = \int_0^L (\frac{\partial \kappa}{\partial s}(s)^2 + \kappa^2(s)\tau^2(s) - \frac{1}{4}\kappa^4(s))\,ds.$$

Functions f_0, f_1, f_2 are in involution, i.e., they Poisson commute pairwise. There is a hierarchy of functions that contains f_0, f_1, f_2 such that any two functions in the hierarchy Poisson commute. For instance, D.Krepski has shown that $f_3 = \int_0^L \tau(s)\,ds$ is in this hierarchy, and he has also shown that the flow of the corresponding Hamiltonian vector field generates the curve shortening equation ([4])

$$\frac{\partial \Lambda}{\partial t}(s,t) = \frac{\partial \Lambda}{\partial s}(s,t) = \kappa(s,t)N(s,t).$$

A detailed investigation of this hierarchy of Poisson commuting functions and its relation to the hierarchies obtained by Langer-Perline and Shabat-Zacharov will be deferred to a separate study.

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